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# An analytical solution of the S-model kinetic equations

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**Abstract.** An analytical variation of the discrete ordinates method is used to establish a concise and accurate solution to the plane Poiseuille and thermal creep problems based on the S-model kinetic equations. The results are of high accuracy and include the flow rates and the heat fluxes of both problems for a wide range of Knudsen numbers and various values of the accommodation coefficient at the boundaries.

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## 1. Introduction

The plane Poiseuille and thermal creep flows are two of the classical problems in rarefied gas dynamics and have been solved, over the years, with various analytical and numerical methods [1,2]. The numerical approaches are based on the Boltzmann equation or simplified kinetic equations including the BGK model [3], the S-model [4] and variable collision frequency models. Today with the help of powerful computers it is possible to solve numerically complex geometry problems, supplemented by advanced kinetic models. The analytical results however, in most cases are limited to simple geometries and models. It is always advisable to compare the numerical results with analytical solutions in order to justify the accuracy to expect from the computational schemes. This is not always possible, since most of the available exact analysis is based only on the BGK model. It seems necessary to extend the existing analytical approaches and develop new ones in order to face analytically more challenging problems.

Recently an analytical variation of the discrete-ordinates method has been developed [5-8] to establish particularly concise and accurate solutions to a number of classical kinetic theory problems. All this analysis is based again on BGK type equations. In the present work an attempt is made to extend this novel analytical approach to more sophisticated kinetic models. The plane Poiseuille and thermal creep problems are solved, based on the S-model kinetic equations.

The S-model has been extensively used in the past [9] and also recently [10-12] to describe efficiently, at the same time, isothermal and non-isothermal flows. It is known that in these cases the classical BGK can not provide simultaneously accurate results and yields erroneous Prandtl numbers. Typical examples are the Poiseuille and thermal creep problems, where the produced mass and heat fluxes are due to the combined effect of the imposed pressure and temperature gradients. Most of the numerical work with the S-model is based on the discrete velocity method [9-12]. A thorough review of all related work in internal rarefied gas flows is given by Sharipov [2]. The results of the present work allow a validation of the existing numerical results and also demonstrate the ability of the new analytical approach to solve accurately more complicated kinetic models.

The linearized S-model kinetic equations [2,10] for the plane Poiseuille and thermal creep problems are described by

$$c_y \frac{\partial h_{\mathrm{P}}(y,\boldsymbol{c})}{\partial y} = \frac{1}{\theta}[2u_{\mathrm{P}}c_x + \frac{4}{15}q_{\mathrm{P}}c_x(\boldsymbol{c}^2 - \frac{5}{2}) - h_{\mathrm{P}}] - c_x \tag{1}$$

and

$$c_y \frac{\partial h_{\mathrm{T}}(y,\boldsymbol{c})}{\partial y} = \frac{1}{\theta}[2u_{\mathrm{T}}c_x + \frac{4}{15}q_{\mathrm{T}}c_x(\boldsymbol{c}^2 - \frac{5}{2}) - h_{\mathrm{T}}] - c_x(\boldsymbol{c}^2 - \frac{5}{2}) \tag{2}$$

respectively. In the above equations $x$ denotes the longitudinal direction and $y$ the vertical direction, $\boldsymbol{c}$ is the molecular velocity vector and the rarefaction parameter $\theta$ is the Knudsen number. The bulk velocity $u(y)$ and the heat flow $q(y)$ are expressed via the unknown linearized distribution functions $h_{\mathrm{P}}(y,\boldsymbol{c})$ and $h_{\mathrm{T}}(y,\boldsymbol{c})$ as

$$u_i(y) = \pi^{-3/2} \int_{-\infty}^{\infty} h_i(y,\boldsymbol{c}) c_x \mathrm{e}^{-\boldsymbol{c}^2}\, \mathrm{d}\boldsymbol{c} \tag{3}$$

and

$$q_i(y) = \pi^{-3/2} \int_{-\infty}^{\infty} h_i(y,\boldsymbol{c}) c_x (\boldsymbol{c}^2 - \frac{5}{2}) \mathrm{e}^{-\boldsymbol{c}^2}\, \mathrm{d}\boldsymbol{c}, \tag{4}$$

where $i = \mathrm{P}$ for the Poiseuille problem and $i = \mathrm{T}$ for the thermal creep problem. When the bulk velocity and the heat flow are integrated over the distance between the two plates, yield the flow rates $\mathrm{V_P}$, $\mathrm{V_T}$ and the heat fluxes $Q_{\mathrm{P}}$, $Q_{\mathrm{T}}$, due to the pressure and temperature gradients respectively. If only the flow rates are to be computed, then Eq. (2) need not be solved, since the well known Onsager's relation [13,14]

$$\mathrm{V_T} = Q_{\mathrm{P}} \tag{5}$$

may be used. In this work both Eqs (1) and (2) are solved in order to have a complete solution and also to test the precision of the implemented semi analytical numerical scheme satisfying Eq. (5).

## 2. Formulation of the Poiseuille and thermal creep problems

The mathematical manipulation of Eqs (1) and (2) is similar. For that reason the detailed formulation of the method is presented only for Eq. (1). To eliminate the variables $c_x$ and $c_z$, two new functions

$$\mathrm{Z}_1^{\mathrm{P}}(y, c_y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{\mathrm{P}}(y, c_x', c_y', c_z') c_x \mathrm{e}^{-{c_x'}^2 - {c_z'}^2} \, \mathrm{d}c_x' \, \mathrm{d}c_z'. \tag{6}$$

and

$$\mathrm{Z}_2^{\mathrm{P}}(y, c_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{\mathrm{P}}(y, c_x', c_y', c_z') c_x (c_x^2 + c_z^2) \mathrm{e}^{-{c_x'}^2 - {c_z'}^2} \, \mathrm{d}c_x' \, \mathrm{d}c_z', \tag{7}$$

where the P denotes the Poiseuille problem, are introduced. Equation (1) is multiplied by $c_x \mathrm{e}^{-{c_x'}^2 - {c_z'}^2}$ and $c_x(c_x^2 + c_z^2)\mathrm{e}^{-{c_x'}^2 - {c_z'}^2}$ successively and it is integrated accordingly, to deduce the following two coupled equations:

$$\frac{\theta}{2} + \theta c_y \frac{\partial \mathrm{Z}_1^{\mathrm{P}}(y, c_y)}{\partial y} + \mathrm{Z}_1^{\mathrm{P}}(y, c_y) = u_{\mathrm{P}}(y) + \frac{2}{15}(c_y^2 - \frac{1}{2}) q_{\mathrm{P}}(y), \tag{8}$$

$$\frac{\theta}{2} + \theta c_y \frac{\partial \mathrm{Z}_2^{\mathrm{P}}(y, c_y)}{\partial y} + \mathrm{Z}_2^{\mathrm{P}}(y, c_y) = u_{\mathrm{P}}(y) + \frac{2}{15}(c_y^2 + \frac{1}{2}) q_{\mathrm{P}}(y). \tag{9}$$

The quantities $u_{\mathrm{P}}(y)$ and $q_{\mathrm{P}}(y)$ in terms of the two new functions are defined by

$$u_{\mathrm{P}}(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Z}_1^{\mathrm{P}}(y, c_y') \mathrm{e}^{-{c_y'}^2} \, \mathrm{d}c_y' \tag{10}$$

and

$$q_{\mathrm{P}}(y) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} ({c_y'}^2 - \frac{5}{2}) \mathrm{Z}_1^{\mathrm{P}}(y, c_y') \mathrm{e}^{-{c_y'}^2} \, \mathrm{d}c_y' + \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mathrm{Z}_2^{\mathrm{P}}(y, c_y') \mathrm{e}^{-{c_y'}^2} \, \mathrm{d}c_y'. \tag{11}$$

If $d$ is the distance between the plates, the boundary conditions for $\mathrm{Z}_1^{\mathrm{P}}(y, c_y)$ and $\mathrm{Z}_2^{\mathrm{P}}(y, c_y)$ at the walls, for molecules leaving the wall, are given as

$$\mathrm{Z}_i^{\mathrm{P}}[\mp d/2, \pm c_y] = (1 - \alpha) \mathrm{Z}_i^{\mathrm{P}}[\mp d/2, \mp c_y], \tag{12}$$

for i=1,2, where $\alpha$ is the accommodation coefficient at the boundaries.

To continue our formulation it is essential at this point to find the particular solutions of Eqs (8) and (9). To simplify notation let $\xi = c_y, \tau = y/\theta$ and $\delta = d/2\theta$, where $\delta$ is half the distance between the plates. It is seen that

$$Y_{1p}^{\mathrm{P}} = \frac{1}{2}\theta(\tau^2 - 2\tau\xi + \frac{11}{5}\xi^2 - \delta^2) \tag{13}$$

and

$$Y_{2p}^{\mathrm{P}} = \frac{1}{2}\theta(\tau^2 - 2\tau\xi + \frac{11}{5}\xi^2 + \frac{1}{5} - \delta^2), \tag{14}$$

are particular solutions of Eqs (8) and (9). Then $\mathrm{Z}_i^{\mathrm{P}}(x, \xi)$, for $i = 1, 2$ can be written as

$$\mathrm{Z}_i^{\mathrm{P}}(\tau, \xi) = Y_{ip}^{\mathrm{P}}(\tau, \xi) - \theta Y_i^{\mathrm{P}}(\tau, \xi). \tag{15}$$

The unkown functions $Y_i^{\mathrm{P}}$ satisfy the homogeneous vector equation

$$\xi \frac{\partial \boldsymbol{Y}^{\mathrm{P}}(\tau,\xi)}{\partial \tau} + \boldsymbol{Y}^{\mathrm{P}}(\tau,\xi) = \frac{1}{\sqrt{\pi}} \mathbf{A}(\xi) \int_{-\infty}^{\infty} \mathbf{B}(\xi') \boldsymbol{Y}^{\mathrm{P}}(\tau,\xi') \mathrm{e}^{-\xi'^2}\, \mathrm{d}\xi', \tag{16}$$

where

$$\boldsymbol{Y}^{\mathrm{P}}(x,\xi) = \begin{bmatrix} Y_1^{\mathrm{P}}(x,\xi) \\ Y_2^{\mathrm{P}}(x,\xi) \end{bmatrix}, \tag{17}$$

$$\mathbf{A}(\xi) = \begin{bmatrix} 1 & (2/15)\big(\xi^2 - 1/2\big) \\ 1 & (2/15)\big(\xi^2 + 1/2\big) \end{bmatrix} \tag{18}$$

and

$$\mathbf{B}(\xi') = \begin{bmatrix} 1 & 0 \\ (\xi'^2 - 5/2) & 2 \end{bmatrix}. \tag{19}$$

The associated boundary conditions at $\tau = \mp\delta$ are

$$Y_1^{\mathrm{P}}(\mp\delta, \pm\xi) = (1-\alpha) Y_1^{\mathrm{P}}(\mp\delta, \mp\xi) + (2-\alpha)\delta\xi + \alpha \frac{11}{10}\xi^2 \tag{20}$$

and

$$Y_2^{\mathrm{P}}(\mp\delta, \pm\xi) = (1-\alpha) Y_2^{\mathrm{P}}(\mp\delta, \mp\xi) + (2-\alpha)\delta\xi + \alpha \frac{11}{10}\xi^2 + \alpha\frac{1}{10}, \tag{21}$$

for $\xi > 0$. Substituting Eq. (15) into Eqs (10) and (11), the velocity and the heat flow profiles for the Poiseuille problem may be expressed in non dimensional form as

$$u_{\mathrm{P}}(\tau) = \frac{1}{2}\Big(\frac{11}{10} - \alpha^2 + \tau^2\Big) - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} Y_1^{\mathrm{P}}(\tau,\xi') \mathrm{e}^{-\xi'^2}\, \mathrm{d}\xi' \tag{22}$$

and

$$q_{\mathrm{P}}(\tau) = \frac{3}{4} - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \Big(\xi^2 - \frac{5}{2}\Big) Y_1^{\mathrm{P}}(x,\xi') \mathrm{e}^{-\xi'^2}\, \mathrm{d}\xi' - \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} Y_2^{\mathrm{P}}(x,\xi') \mathrm{e}^{-\xi'^2}\, \mathrm{d}\xi'. \tag{23}$$

Operating similarly on Eq. (2) the thermal creep problem is formulated and the corresponding expressions for the bulk velocity and heat flow profiles, due to the imposed temperature gradient, are found. In this case the particular solutions are

$$Y_{1p}^{\mathrm{T}} = \frac{3}{4}\theta\Big(\xi^2 - \frac{1}{2}\Big) \tag{24}$$

and

$$Y_{2p}^{\mathrm{T}} = \frac{3}{4}\theta\Big(\xi^2 + \frac{1}{2}\Big), \tag{25}$$

where T denotes the thermal creep problem. The homogeneous part of the solution $Y_i^{\mathrm{T}}$ satisfies the same vector equation (16), coupled with the boundary conditions

$$Y_1^{\mathrm{T}}(\mp\delta, \pm\xi) = (1-\alpha) Y_1^{\mathrm{T}}(\mp\delta, \mp\xi) + \alpha\frac{3}{4}\Big(\xi^2 - \frac{1}{2}\Big) \tag{26}$$

and

$$Y_2^{\mathrm{T}}(\mp\delta, \pm\xi) = (1-\alpha) Y_2^{\mathrm{T}}(\mp\delta, \mp\xi) + \alpha \frac{3}{4}(\xi^2 + \frac{1}{2}). \tag{27}$$

Finally the quantities of main interest, which are the flow rates and the heat fluxes, are obtained from the integrals

$$\mathrm{V}_i(\delta) = -\frac{1}{2\delta^2} \int_{-\delta}^{\delta} u_i(\tau)\, \mathrm{d}\tau. \tag{28}$$

$$Q_i(\delta) = -\frac{1}{2\delta^2} \int_{-\delta}^{\delta} q_i(\tau)\, \mathrm{d}\tau. \tag{29}$$

respectively, where $i = \mathrm{P}$ for the Poiseuille problem and $i = \mathrm{T}$ for the thermal creep problem. In section 4, results of the above quantities are presented as function of the rarefunction parameter and the accommodation coefficient.

## 3. The discrete-ordinates solution

The analytical version of the classical discrete ordinate (or discrete velocity) method has already been implemented, based on the BGK model, in a series of classical rarefied gas dynamics problems [5-8]. For that reason our discussion here is brief, pointing out the new elements of the formulation related to the S-model kinetic equations.

The problem has been reduced to the solution of Eq. (16), subject to the boundary conditions (20), (21) and (26), (27) for the Poiseuille and thermal creep problems respectively. Since Eq. (16) is common the superscript notation is not used and it is assumed that it corresponds to both problems. Assuming that Eq. (16) holds for a finite number of discrete velocities $\xi_i$, its discrete ordinate approximation may be written as

$$\pm\xi_i \frac{\mathrm{d}}{\mathrm{d}x} \boldsymbol{Y}(x, \pm\xi_i) + \boldsymbol{Y}(x, \pm\xi_i) = \mathbf{A}(\xi_i) \sum_{k=1}^{N} \hat{w}_k \mathbf{B}(\xi_k) \big[\boldsymbol{Y}(x, \xi_k) + \boldsymbol{Y}(x, -\xi_k)\big], \tag{30}$$

where

$$\hat{w}_k = \pi^{-1/2} w_k \mathrm{e}^{-\xi_k^2}, \tag{31}$$

for $i = 1, 2, \ldots, N$, while a compatible quadrature scheme is applied to the integral terms. Following the main idea of the method of elementary solutions [14], the complete set of separable solutions

$$\boldsymbol{Y}(x, \pm\xi_i) = \boldsymbol{\Phi}(\nu, \pm\xi_i) \mathrm{e}^{-x/\nu}, \tag{32}$$

is substituted into Eq. (30) to find, after some algebraic manipulation,

$$\big(\mathbf{D} - 2\mathbf{M}^{-1}\mathbf{S}\mathbf{W}\mathbf{M}^{-1}\big)\mathbf{M}\big[\boldsymbol{\Phi}_+(\nu) + \boldsymbol{\Phi}_-(\nu)\big] = \lambda \mathbf{M}\big[\boldsymbol{\Phi}_+(\nu) + \boldsymbol{\Phi}_-(\nu)\big], \tag{33}$$

for $i = 1, 2, \ldots, N$. In the eigenvalue problem defined by Eq. (33), $\nu$ are the separation constants, $\lambda = 1/\nu^2$ are the eigenvalues,

$$\mathbf{D} = \mathrm{diag}\big\{ \ldots, \xi_i^{-2}\mathbf{I}, \ldots \big\}, \tag{34}$$

$$\mathbf{M} = \mathrm{diag}\big\{ \ldots, \xi_i\mathbf{I}, \ldots \big\} \tag{35}$$

and

$$\mathbf{S} = \mathrm{diag}\big\{ \ldots, \mathbf{A}(\xi_i), \ldots \big\}, \tag{36}$$

are $2N \times 2N$ block diagonal matrices. The $2N \times 2N$ matrix $\mathbf{W}$ has $N$ block rows each given by

$$\mathbf{R} = \big[\hat{w}_1\mathbf{B}(\xi_1)\ \hat{w}_2\mathbf{B}(\xi_2)\ \cdots\ \hat{w}_N\mathbf{B}(\xi_N)\big] \tag{37}$$

and the vectors

$$\boldsymbol{\Phi}_{\pm}(\nu) = \big[\boldsymbol{\Phi}^{\mathrm{T}}(\nu, \pm\xi_1)\ \boldsymbol{\Phi}^{\mathrm{T}}(\nu, \pm\xi_2)\ \cdots\ \boldsymbol{\Phi}^{\mathrm{T}}(\nu, \pm\xi_N)\big]^{\mathrm{T}}. \tag{38}$$

The eigenvalue problem is solved to obtain the $2N$ eigenvalues and the corresponding $2N$ (positive) separation constants. Then from Eqs (30) and (32) is deduced that

$$\boldsymbol{\Phi}(\nu_j, \pm\xi_i) = \nu_j\big(\nu_j \mp \xi_i\big)^{-1}\mathbf{A}(\xi_i)\mathbf{F}(\nu_j), \tag{39}$$

where

$$\mathbf{F}(\nu_j) = \sum_{k=1}^{N} \hat{w}_k\mathbf{B}(\xi_k)\big[\boldsymbol{\Phi}(\nu_j, \xi_k) + \boldsymbol{\Phi}(\nu_j, -\xi_k)\big]. \tag{40}$$

Equations (39) are multiplied by $\hat{w}_i\mathbf{B}(\xi_i)$ and the resulting equations are summed over $i$ to yield

$$\boldsymbol{\Omega}(\nu_j)\mathbf{F}(\nu_j) = \mathbf{0}, \tag{41}$$

with

$$\boldsymbol{\Omega}(\nu_j) = \mathbf{I} - 2\nu_j^2 \sum_{i=1}^{N} \hat{w}_i\mathbf{B}(\xi_i)\mathrm{diag}\Big\{\frac{1}{\nu_j^2 - \xi_i^2}, \frac{1}{\nu_j^2 - \xi_i^2}\Big\}\mathbf{A}(\xi_i). \tag{42}$$

Here $\mathbf{F}(\nu_j)$ is a vector in the null space of $\boldsymbol{\Omega}(\nu_j)$. It is seen that after the separation constants $\nu_j$ are determined and $\boldsymbol{\Omega}(\nu_j)$ is defined, the vector $\mathbf{F}(\nu_j)$ can be easily computed.

A first version of the analytically deduced discrete ordinates solution of Eq. (16) may be written as

$$\boldsymbol{Y}(x, \pm\xi_i) = \sum_{j=1}^{2N} \nu_j\Big[\frac{\mathrm{A}_j}{\nu_j \mp \xi_i}\mathrm{e}^{-(\alpha+x)/\nu_j} + \frac{\mathrm{B}_j}{\nu_j \pm \xi_i}\mathrm{e}^{-(\alpha-x)\nu_j}\Big]\mathbf{A}(\xi_i)\mathbf{F}(\nu_j), \tag{43}$$

where the $4N$ constants $\{\mathrm{A}_j\}$ and $\{\mathrm{B}_j\}$ are, still arbitrary. Since the determinant of $\boldsymbol{\Omega}(\nu_j)$ has a first order zero at infinity we ignore the contribution in Eq. (43) from the largest separation constant and instead we include the easily deduced exact solution. In this case the modified version of the analytical solution is

$$\boldsymbol{Y}(x,\pm\xi_i)=\boldsymbol{Y}_0(x,\pm\xi_i)+\sum_{j=2}^{2N}\nu_j\big[\frac{\mathrm{A}_j}{\nu_j\mp\xi_i}\mathrm{e}^{-(\alpha+x)/\nu_j}+\frac{\mathrm{B}_j}{\nu_j\pm\xi_i}\mathrm{e}^{-(\alpha-x)\nu_j}\big]\mathbf{A}(\xi_i)\mathbf{F}(\nu_j), \tag{44}$$

where

$$\boldsymbol{Y}_0(x,\pm\xi_i)=\big[\mathrm{A}+\mathrm{B}\big(x,\mp\xi_i\big)\big]\begin{bmatrix}1\\1\end{bmatrix}. \tag{45}$$

The final step is the estimation of the coefficients $\mathrm{A}_j$ and $\mathrm{B}_j, j=1,2,...2N$. They are found by substituting the solution (44) into the boundary conditions (20) and (21) for the Poiseuille problem and into the boundary conditions (26) and (27) for the thermal creep problem. For each problem the resulting algebraic system of $4N$ equations is solved for the $4N$ unknown constants. Thus the two systems are identical except for the right hand sides. Let the coefficients $\mathrm{A}_j^{\mathrm{P}},\mathrm{B}_j^{\mathrm{P}}$ and $\mathrm{A}_j^{\mathrm{T}},\mathrm{B}_j^{\mathrm{T}}, j=1,2,...2N$ denote the solution of the Poiseuille and thermal creep problems respectively.

When Eq. (44) is substituted back into Eqs (22) and (23) the resulting expressions for the local quantities of the velocity and heat flow profiles are

$$u_{\mathrm{P}}(x)=\frac{1}{2}\big(\frac{11}{10}-\delta^2+x^2\big)-\mathrm{A}^{\mathrm{P}}-\mathrm{B}^{\mathrm{P}}x-\sum_{j=2}^{2N}\big(\mathrm{A}_j^{\mathrm{P}}\mathrm{e}^{-(\delta+x)/\nu_j}+\mathrm{B}_j^{\mathrm{P}}\mathrm{e}^{-(\delta-x)\nu_j}\big]f_1(\nu_j) \tag{46}$$

and

$$q_{\mathrm{P}}(x)=-\frac{15}{8}+\sum_{j=2}^{2N}\big(\mathrm{A}_j^{\mathrm{P}}\mathrm{e}^{-(\delta+x)/\nu_j}+\mathrm{B}_j^{\mathrm{P}}\mathrm{e}^{-(\delta-x)\nu_j}\big]f_2(\nu_j) \tag{47}$$

respectively. These distributions are integrated over the distance between the two plates according to Eqs (28) and (29), to obtain the following closed form expressions for the overall quantities of the flow rate

$$\mathrm{V}_{\mathrm{P}}(\delta)=\frac{\delta}{3}-\frac{11}{20\delta}+\frac{\mathrm{A}^{\mathrm{P}}}{\delta}+\frac{1}{2\delta^2}\sum_{j=2}^{2N}\big(\mathrm{A}_j^{\mathrm{P}}+\mathrm{B}_j^{\mathrm{P}}\big)\nu_j\big[1-\mathrm{e}^{-2\delta/\nu_j}\big]f_1(\nu_j), \tag{48}$$

and the heat flux

$$Q_{\mathrm{P}}(\delta)=\frac{3}{4\delta}-\frac{1}{2\delta^2}\sum_{j=2}^{2N}\big(\mathrm{A}_j^{\mathrm{P}}+\mathrm{B}_j^{\mathrm{P}}\big)\nu_j\big[1-\mathrm{e}^{-2\delta/\nu_j}\big]f_2(\nu_j), \tag{49}$$

for the Poiseuille problem. The corresponding expressions for the thermal creep problem are

$$u_{\mathrm{T}}(x)=\mathrm{A}^{\mathrm{T}}+\mathrm{B}^{\mathrm{T}}x+\sum_{j=2}^{2N}\big(\mathrm{A}_j^{\mathrm{T}}\mathrm{e}^{-(\delta+x)/\nu_j}+\mathrm{B}_j^{\mathrm{T}}\mathrm{e}^{-(\delta-x)\nu_j}\big]f_1(\nu_j) \tag{50}$$

and

$$q_{\mathrm{T}}(x) = -\frac{15}{8} + \sum_{j=2}^{2N} \left(\mathrm{A}_j^{\mathrm{T}} \mathrm{e}^{-(\delta+x)/\nu_j} + \mathrm{B}_j^{\mathrm{T}} \mathrm{e}^{-(\delta-x)\nu_j}\right] f_2(\nu_j) \tag{51}$$

for the local quantities and

$$\mathrm{V}_{\mathrm{T}}(\delta) = \frac{\mathrm{A}^{\mathrm{T}}}{\delta} + \frac{1}{2\delta^2} \sum_{j=2}^{2N} \left(\mathrm{A}_j^{\mathrm{T}} + \mathrm{B}_j^{\mathrm{T}}\right) \nu_j \left[1 - \mathrm{e}^{-2\delta/\nu_j}\right] f_1(\nu_j) \tag{52}$$

and

$$Q_{\mathrm{T}}(\delta) = \frac{15}{8\delta} - \frac{1}{2\delta^2} \sum_{j=2}^{2N} \left(\mathrm{A}_j^{\mathrm{T}} + \mathrm{B}_j^{\mathrm{T}}\right) \nu_j \left[1 - \mathrm{e}^{-2\delta/\nu_j}\right] f_2(\nu_j). \tag{53}$$

for the gross quantities.

The method is considered to be a semi-analytical-numerical approach since the only numerical work required, is related with the estimation of the eigenvalues and the unknown coefficients via the solution of two linear systems.

## 4. Numerical results

To solve for the unknown coefficients first a set of collocation points must be selected. Following previous work [5-8], it is chosen to map the initial interval $[0, \infty)$ of $\xi_i$ onto the new interval $[0, 1]$. Then a Gauss-Legendre scheme, mapped also onto the interval $[0, 1]$, is used. This quadrature approximation has been found to be very effective. After the quadrature points have been selected the numerical implementation is simple and straightforward. The only pitfall, which may rise, is when the term $\hat{w}_k$, defined by Eq. (31), for some values of $\xi_k$ becomes zero from a computational point of view. Then some of the separation constants $\nu_j$ will be equal to some of the quadrature nodes $\xi_k$, and this clearly is not allowed in Eq. (43). This problem is circumvented by simply omitting the quadrature points which cause the singularity [5-8]. This is numerically justified since, from a computational point of view, these nodes do not contribute to the right hand side of Eq. (30). At the same time the number of separation constants is reduced accordingly. After the linear systems are solved and all coefficients are found the quantities of main interest are computed from expressions (50–53).

Since there are no exact results available in the literature to compare, it is not possible to prove the accuracy of our solution. The confidence in our results is established by increasing the value of $N$ until convergence to the last digit shown is achieved. Onsager's relation (5) is also used. In addition the numerical solution of the linear systems is obtained using independently developed Fortran and Matlab routines. All presented results are originated with $40 < N < 80$. We believe that the results are accurate to all significant figures shown. The required computational time in a typical PC is 2-3 seconds.

Table 1. Flow rates V and heat fluxes $Q$ for the Poiseuille and thermal creep problems based on the S-model for purely diffuse scattering.

| | *Poiseuille problem* | | *Thermal creep problem* | | *Ref.* [2] *Direct numerical method* | | |
|---|---|---|---|---|---|---|---|
| $2\delta$ | $V_P$ | $Q_P$ | $V_T$ | $Q_T$ | $V_P$ | $Q_P$ *or* $V_T$ | $Q_T$ |
| 0.01 | 3.0517 | 1.2469 | 1.2469 | 6.7341 | 3.0519 | 1.2470 | 6.7343 |
| 0.05 | 2.30720 | 0.87260 | 0.87260 | 4.83114 | - | - | - |
| 0.1 | 2.03955 | 0.73268 | 0.73268 | 4.05461 | 2.0397 | 0.7328 | 4.0553 |
| 0.5 | 1.61445 | 0.46294 | 0.46294 | 2.39170 | 1.6147 | - | - |
| 1 | 1.55365 | 0.36546 | 0.36546 | 1.75375 | 1.5541 | 0.3656 | 1.7543 |
| 5 | 2.00543 | 0.16399 | 0.16399 | 0.61667 | 2.0080 | - | - |
| 10 | 2.77990 | 0.098147 | 0.098147 | 0.34063 | 2.7863 | 0.09834 | 0.3407 |
| 100 | 17.69675 | 0.0115516 | 0.0115516 | 0.037155 | - | - | - |

Table 2. Flow rates $V_P$ for the Poiseuille problem based on the S-model and the Boltzmann equation (BE) for diffuse-specular scattering.

| | $\alpha = 1.0$ | | $\alpha = 0.75$ | | $\alpha = 0.50$ | |
|---|---|---|---|---|---|---|
| $2\delta$ | S *model* | BE[16] | S *model* | BE[17] | S *model* | BE[17] |
| 0.01 | 3.0517 | - | 4.5319 | - | 7.2100 | - |
| 0.05 | 2.30720 | - | 3.35072 | - | 5.24276 | - |
| 0.1 | 2.03955 | 1.9318 | 2.93646 | 2.7860 | 4.58009 | 4.3628 |
| 0.5 | 1.61445 | 1.5607 | 2.28417 | 2.2128 | 3.57177 | 3.4748 |
| 1.0 | 1.55365 | 1.5086 | 2.17393 | 2.1204 | 3.39277 | 3.3270 |
| 5.0 | 2.00543 | 1.9637 | 2.60139 | 2.5555 | 3.78842 | 3.7496 |
| 10.0 | 2.77990 | 2.7350 | 3.38599 | 3.3407 | 4.58372 | 4.5490 |
| 100.0 | 17.69675 | - | 18.32685 | - | 19.53951 | - |

In Table 1 the flow rates and the heat fluxes are presented for different values of the inverse Knudsen number $2\delta$, which represents the distance between the plates, with diffuse boundary conditions for the Poiseuille and thermal creep problems. In columns 3 and 4 the heat flux $Q_P$ and the flow rate $V_T$ are given. It is seen that Onsager's relation (5) is satisfied very accurately. In the last three columns of the same table relative results of previous work (see Tables 1, 11 and 18 of Ref. [2]), using the S-model and a numerical discrete velocity method, are also

Table 3. Heat fluxes $Q_{\mathrm{P}}$ and flow rates $\mathrm{V}_{\mathrm{T}}$ for the Poiseuille and thermal creep problems respectively, based on the S-model and the Boltzmann equation (BE) for diffuse-specular scattering.

| | $\alpha = 1.0$ | | $\alpha = 0.75$ | | $\alpha = 0.50$ | |
|---|---|---|---|---|---|---|
| $2\delta$ | S *model* | BE[17] | S *model* | BE[17] | S *model* | BE[17] |
| 0.01 | 1.2469 | - | 1.8020 | - | 2.7706 | - |
| 0.05 | 0.87260 | - | 1.20557 | - | 1.76508 | - |
| 0.1 | 0.73268 | 0.7966 | 0.98589 | 1.0864 | 1.40121 | 1.5632 |
| 0.5 | 0.46294 | 0.5036 | 0.56581 | 0.6225 | 0.71551 | 0.7903 |
| 1.0 | 0.36546 | 0.3890 | 0.41916 | 0.4505 | 0.49043 | 0.5285 |
| 5.0 | 0.16399 | 0.1574 | 0.16112 | 0.1570 | 0.15787 | 0.1566 |
| 10.0 | 0.098147 | 0.0898 | 0.093048 | 0.0871 | 0.087524 | 0.0842 |
| 100.0 | 0.011552 | - | 0.010643 | - | 0.0096702 | - |

Table 4. Heat fluxes $Q_{\mathrm{T}}$ for the thermal creep problem based on the S-model and the Boltzmann equation (BE) for diffuse-specular scattering.

| | $\alpha = 1.0$ | | $\alpha = 0.75$ | | $\alpha = 0.50$ | |
|---|---|---|---|---|---|---|
| $2\delta$ | S *model* | BE[16] | S *model* | BE[17] | S *model* | BE[17] |
| 0.01 | 6.7341 | - | 9.9081 | - | 15.5042 | - |
| 0.05 | 4.83114 | - | 6.80590 | - | 10.10061 | - |
| 0.1 | 4.05461 | 3.8669 | 5.55960 | 5.3371 | 7.97685 | 7.7430 |
| 0.5 | 2.39170 | 2.3918 | 2.98291 | 2.9969 | 3.79898 | 3.8420 |
| 1.0 | 1.75375 | 1.7846 | 2.06847 | 2.1077 | 2.46400 | 2.5136 |
| 5.0 | 0.61667 | 0.6319 | 0.64682 | 0.6602 | 0.67894 | 0.6903 |
| 10.0 | 0.34063 | 0.3467 | 0.34859 | 0.3540 | 0.35694 | 0.3616 |
| 100.0 | 0.037155 | - | 0.037235 | - | 0.037319 | - |

presented. The numerical results given in Ref. [2] are in good agreement with the analytical results.

In Tables 2 , 3 and 4 the gross quantities $\mathrm{V}_{\mathrm{P}}$ , $\mathrm{V}_{\mathrm{T}}$ (or $Q_{\mathrm{P}}$ ) and $Q_{\mathrm{T}}$ are given respectively, for different accommodation coefficients $\alpha$ and various distances between the plates. The corresponding results of Refs [16] and [17], obtained by a discrete velocity numerical solution of the Boltzmann equation are also presented. It is seen that the obtained results of the S-model and the Boltzmann equation are

Table 5. Velocity and heat flow profiles for the Poiseuille and thermal creep problems based on the S-model for $\alpha = 0.75$ and $2\delta = 0.1$.

| $2\delta$ | $-u_{\rm P}(x)$ | $q_{\rm P}(x)$ | $u_{\rm T}(x)$ | $-q_{\rm T}(x)$ |
|---|---|---|---|---|
| 0.0 | 0.15229 | 0.051946 | 0.051807 | 0.28808 |
| 0.2 | 0.15171 | 0.051669 | 0.051544 | 0.28702 |
| 0.4 | 0.14994 | 0.050814 | 0.050734 | 0.28377 |
| 0.6 | 0.14680 | 0.049297 | 0.049296 | 0.27799 |
| 0.8 | 0.14187 | 0.046895 | 0.047020 | 0.26884 |
| 1.0 | 0.13289 | 0.042444 | 0.042812 | 0.25194 |

Table 6. Velocity and heat flow profiles for the Poiseuille and thermal creep problems based on the S-model for $\alpha = 0.75$ and $2\delta = 1.0$.

| $2\delta$ | $-u_{\rm P}(x)$ | $q_{\rm P}(x)$ | $u_{\rm T}(x)$ | $-q_{\rm T}(x)$ |
|---|---|---|---|---|
| 0.0 | 1.1875 | 0.24389 | 0.23414 | 1.1122 |
| 0.2 | 1.1768 | 0.24048 | 0.23166 | 1.1045 |
| 0.4 | 1.1439 | 0.22986 | 0.22398 | 1.0806 |
| 0.6 | 1.0862 | 0.21060 | 0.21012 | 1.0369 |
| 0.8 | 0.99608 | 0.17883 | 0.18750 | 0.96423 |
| 1.0 | 0.83112 | 0.11263 | 0.14182 | 0.81309 |

always in good agreement. The well known capability of the linearized S-model to approximate reasonably well, both the Poiseuille and thermal creep problems simultaneously, is confirmed even for diffuse-specular scattering.

Finally in Tables 5 , 6 and 7 the local quantities of the velocity and heat flow profiles are given for an accommodation coefficient $\alpha = 0.75$ and for typical values

Table 7. Velocity and heat flow profiles for the Poiseuille and thermal creep problems based on the S-model for $\alpha = 0.75$ and $2\delta = 10.0$.

| $2\delta$ | $-u_{\rm P}(x)$ | $q_{\rm P}(x)$ | $u_{\rm T}(x)$ | $-q_{\rm T}(x)$ |
|---|---|---|---|---|
| 0.0 | 21.424 | 0.68318 | 0.52095 | 1.8525 |
| 0.2 | 20.908 | 0.67142 | 0.51772 | 1.8477 |
| 0.4 | 19.351 | 0.62971 | 0.50637 | 1.8300 |
| 0.6 | 16.714 | 0.53211 | 0.48039 | 1.7854 |
| 0.8 | 12.862 | 0.29559 | 0.41984 | 1.6641 |
| 1.0 | 6.5128 | -0.72232 | 0.19532 | 1.0624 |

of the distance between the plates equal to $0.1$ , $1.0$ and $10.0$ . The change of sign in the heat flow profile near the wall as the distance between the plates is increased, is due to the influence of the surface and it is expected. More important it is noticed that the accuracy to expect of the method remains the same either computing gross or local quantities. Actually the estimation of all relative quantities, as it is seen from Eqs (50-53), including the distribution function itself, is based on the determination of the same $4N$ unknown coefficients. This is one of the advantages of the present semi analytical approach.

## 5. Concluding remarks

The plane Poiseuille and thermal creep problems, described by the S-model kinetic equations have been solved, based on a semi analytical numerical version of the discrete ordinates method. This analytical approach is extended for first time to a collision model other than the BGK. The implementation however, remains very simple and the method is easy to use. The obtained results are of high accuracy and allow the reliable testing of previous results obtained with direct numerical approaches, based on the S-model approximation.

Following the present work, we are more optimistic now than before, about applying this novel approach to other geometries and to even more sophisticated kinetic models, including the variable frequency collision model.

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