Plane stress elastoplastic solutions of interface cracks with contact zones

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An elastoplastic solution for the plane stress problem of an interface crack with contact zones is presented. The problem of a crack lying along the interface of an elastoplastic medium and a rigid substrate is analyzed in detail. Any contact between the crack surface and the substrate is assumed to be frictionless. The $J_2$ deformation theory with power-law hardening is used to describe the constitutive behavior of the deformable medium and a two-term elastoplastic asymptotic solution is developed. Large scale yielding finite element solutions for a Griffith interface crack with contact zones are also presented. The numerical solutions are used to verify the elastoplastic solutions developed. The possible application of the rigid substrate solutions to the problem of a crack along the interface of two deformable media is also discussed.

1. Introduction

Elastic analysis of cracks at bimaterial interfaces have focused primarily on two approaches. The traditional approach assumes that the crack faces are traction free (Williams, 1959; Erdogan, 1963; Salganik, 1963; England, 1965; Erdogan, 1965; Rice and Sih, 1965) which leads to solutions with oscillatory stress singularities. These solutions predict wrinkling of crack faces and overlapping of materials, which contradicts the assumption that the crack faces are traction free (England, 1965; Malyshev and Salganik, 1965).

Alternative solutions have been presented by Comninou (1977a, 1978) and Comninou and Schmueser (1979) who examined the possibility of frictionless contact along the crack face. For a Griffith crack along the interface of two dissimilar linear elastic media, these solution predict the existence of two contact zones, one at each tip, the sizes of which depend on the type of far field loading. The stress singularity is of the square root type and the asymptotic solution is mode-II like, in the sense that the shear stress is the only singular stress component ahead of the crack. An analytic solution of the interface crack problem with contact zones was first presented by Gaute-sen and Dundurs (1987, 1988). Hayashi and Nemat-Nasser (1981) presented a solution for the problem of a plane strain branched crack in the interface between two elastic materials; in their solution, a small contact region was introduced in the vicinity of each crack tip in order to remove the oscillatory singularities. More recently, Ni and Nemat-Nasser (1991) have presented analytic solutions for interface cracks in anisotropic dissimilar materials, where the fully-open-crack model and Comninou's contact zone model are analyzed in detail.

In a recent paper, Aravas and Sharma (1991) have shown that the traditional and Comninou solutions do not necessarily exclude each other. They demonstrated that, when the size of the contact zone is a small fraction of the crack length, the Comninou asymptotic solution is the relevant solution sufficiently close to the tip, whereas the traditional open-crack asymptotic solution is still valid at distances larger than the size of the contact zone but still small compared to the crack length. They also showed that while the Comninou solution does not predict interpenetration of the crack faces, it does predict "material interpenetr-
tion” of a different kind – material elements near the crack tip are “turned inside out” and cross the bimaterial interface.

The effects of plastic deformation in the region near the tip of an interfacial plane strain traction-free crack have been considered by Shih and Asaro (1988, 1989), Zywicz (1988), and Zywicz and Parks (1990b). Zywicz and Parks (1990a) have also developed slip line solutions for the near tip region of a plane strain crack with contact zones lying along the interface between a rigid substrate and an elastic–perfectly-plastic material. Aravas and Sharma (1991) presented a two-term asymptotic expansion of the near-tip elastoplastic solution of a plane strain interfacial crack, the faces of which were assumed to be in frictionless contact near the crack tip. Using $J_2$ deformation theory they showed that the asymptotic solution is, to leading order, separable in $r$ and $\theta$, where $(r, \theta)$ are polar coordinates at the crack tip, and that the second term in the stress expansion is a hydrostatic stress term.

In this paper, we consider the plane stress problem of a crack lying along the interface of an elastoplastic medium and a rigid substrate. The possibility of frictionless contact is considered. In Section 2, the Comninou elastic solution is briefly discussed and the prediction of “material interpenetration” is highlighted. A two-term asymptotic elastoplastic solution is developed in Sections 3 and 4. In Section 5, finite element large scale yielding solutions for a Griffith interface crack are presented and comparisons with the predictions of the asymptotic solution are made. The possible application of the rigid substrate solutions to the problem of a crack along the interface of two deformable media is discussed in Section 6.

Standard notation is used throughout. Boldface symbols denote tensors, the order of which is indicated by the context, and the summation convention is used for repeated Latin indices.

2. The linear elastic solution

We consider the two-dimensional plane stress problem of a crack lying along the interface of a homogeneous linear elastic medium and a rigid substrate. The crack face is in frictionless contact with the substrate near the tips. The nature of the elastic fields near the crack tip for the problem considered here can easily be established by William's technique (Williams, 1952). Referring to Fig. 1, the boundary conditions to be satisfied are

$$u_r(r, 0) = 0, \quad u_\theta(r, 0) = 0.$$  \hspace{1cm} (2.1)

and

$$u_\theta(r, \pi) = 0, \quad \sigma_\theta(r, \pi) = 0.$$  \hspace{1cm} (2.2)

The dominant terms in the asymptotic solution for the elastic medium ($0 \leq \theta \leq \pi$) are found to be (Comninou, 1977a)

$$\sigma_{xx} = -\frac{K_{II}}{\sqrt{2\pi r}} \frac{1}{2(\kappa + 1)} \times \left[ (2\kappa + 5) \sin \frac{1}{2}\theta + \sin \frac{3}{2}\theta \right],$$  \hspace{1cm} (2.3)

$$\sigma_{yy} = \frac{K_{II}}{\sqrt{2\pi r}} \frac{1}{2(\kappa + 1)} \left[ (2\kappa - 3) \sin \frac{1}{2}\theta + \sin \frac{3}{2}\theta \right],$$  \hspace{1cm} (2.4)

$$\sigma_{\theta v} = \frac{K_{II}}{\sqrt{2\pi r}} \frac{1}{2(\kappa + 1)} \left[ (2\kappa + 1) \cos \frac{1}{2}\theta + \cos \frac{3}{2}\theta \right].$$  \hspace{1cm} (2.5)

and

$$u_r = \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \frac{1}{(\kappa + 1)} \times \left[ (4\kappa + 1) \sin \frac{1}{2}\theta + \sin \frac{3}{2}\theta \right],$$  \hspace{1cm} (2.6)

$$u_\theta = \frac{K_{II}}{2G} \sqrt{\frac{r}{2\pi}} \frac{1}{(\kappa + 1)} (\cos \frac{1}{2}\theta - \cos \frac{3}{2}\theta).$$  \hspace{1cm} (2.7)

where

$$K_{II} = \lim_{r \to 0} \left( \sqrt{2\pi r \sigma_{yy}} \right)_{\theta = 0}. $$  \hspace{1cm} (2.8)
\( \sigma \) is the stress tensor, \( u \) is the displacement vector, \( G \) is the elastic shear modulus, \( \nu \) is Poisson’s ratio and \( \kappa = (3 - \nu)/(1 + \nu) \) for plane stress. For the above solution to be valid, the stress intensity factor \( K_{1c} \) should be negative so that compressive stresses develop along the contact zone on \( \theta = \pi \). This, in turn, would result in a negative shear stress ahead of the crack, i.e. \( \sigma_{xy}(\theta = 0) < 0 \) as \( r \to 0 \).

Aravas and Sharma (1991) have shown that the above solution predicts material interpenetration in a region near the crack tip. It can be readily shown that \( u_y(x, y) + y \) is negative as \( r \to 0 \), which implies that material points near the crack tip cross the interface and material elements are “turned inside out” in the interpenetration region. Aravas and Sharma (1991) obtained an estimate for the size and shape of the interpenetration region by using the first term in the asymptotic expansion of the solution (i.e. Eq. (2.7)). Consideration of the analytic solution of Gautesen and Dundurs (1987) or inclusion of higher order terms in the asymptotic expansion of the solution will modify the predicted shape of the interpenetration region but will not affect the conclusion of material interpenetration.

In concluding this section we note that the interpenetration is a consequence of the use of linear kinematics in a region where large strains develop. This observation does not invalidate the solution, but, merely indicates that the solution is not relevant in a region very close to the crack tip where finite strains develop. If the region of finite strains is smaller than the size of the predicted contact zone, the Comninou asymptotic solution is still valid over distances that are larger than the finite strain region but still small compared to the size of the contact zone.

The effects of plastic deformation in the near tip region are discussed in detail in the following Sections.

3. Formulation of the nonlinear problem

We consider the plane stress problem of a crack along the interface of a homogeneous isotropic elastoplastic material and a rigid substrate. The crack face is assumed to be in frictionless contact with the rigid surface near the crack tip. The constitutive behavior of the deformable medium is described by the J2 deformation theory for a Ramberg-Osgood uniaxial stress–strain behavior, namely

\[
\epsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} + \frac{1 - 2\nu}{3E} \sigma_{kk} \delta_{ij} + \frac{1}{2} \alpha \epsilon_0 \left( \frac{\sigma_{ij}}{\sigma_0} \right)^{\kappa - 1} \frac{1}{\sigma_0},
\]

where \( \epsilon \) is the infinitesimal strain tensor, \( \sigma \) is the stress tensor and \( s \) its deviatoric part, \( \delta_{ij} \) is the Kronecker delta, \( E \) is Young's modulus, \( \nu \) is Poisson's ratio, \( \alpha \) is a material constant, \( n \) is the hardening exponent \((1 \leq n < \infty)\), \( \sigma_0 \) is the yield stress, \( \epsilon_0 = \sigma_0/E \), and \( \sigma_\epsilon \) is the Mises equivalent stress defined as

\[
\sigma_\epsilon = \left( \frac{1}{2} s_{ij} s_{ij} \right)^{1/2}.
\]

Referring to a polar cylindrical coordinate system on the plane of stressing, the only non-zero stress, strain, and displacement components are \( \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \epsilon_{rr}, \epsilon_{\theta\theta}, \epsilon_{\phi\phi}, \epsilon_{zz}, u_r, u_\theta, \) and \( u_z \), where \( z \) is the coordinate axis normal to the plane of stressing.

The equilibrium equations are written as

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0, \tag{3.3}
\]

\[
\frac{\partial \sigma_{\theta\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + 2\frac{\sigma_{\theta\theta}}{r} = 0. \tag{3.4}
\]

The strain–displacement equations are

\[
\epsilon_{rr} = \frac{\partial u_r}{\partial r},
\]

\[
\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta},
\]

\[
\epsilon_{\phi\phi} = \frac{1}{2} \left( \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right),
\]

and the corresponding compatibility equation is

\[
\left( \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{1}{r} \frac{\partial}{\partial r} \right) \epsilon_{rr} + \left( 2 \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) \epsilon_{\theta\theta} - \left( \frac{2}{r^2} \frac{\partial}{\partial \theta} + \frac{2}{r} \frac{\partial^2}{\partial r \partial \theta} \right) \epsilon_{r\theta} = 0. \tag{3.8}
\]
4. Asymptotic solution procedure

We consider a polar coordinate system in which \( r \) and \( \theta \) are centered at the crack tip (Fig. 1), and look for an asymptotic solution as \( r \to 0 \). In order to determine the terms in the asymptotic solution we adopt the technique developed by Sharma and Aravas (1991). This technique consists of formulating the problem in terms of fundamental quantities, namely stresses and displacements and obtaining a hierarchy of problems which are then solved using a Galerkin-finite-element technique. The procedure is briefly described in what follows.

We attempt an asymptotic expansion of the solution in the form

\[
\frac{\sigma(r, \theta)}{\sigma_0} = r^{s} \sigma^{(0)}(\theta) + r^{s+1} \sigma^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.1)

where \( s < t < \cdots \).

The corresponding expansion for the Mises equivalent stress is

\[
\frac{\sigma_e(r, \theta)}{\sigma_0} = r^{s} \sigma_e^{(0)} + r^{s+1} \sigma_e^{(1)} + \cdots \text{ as } r \to 0,
\]

(4.2)

where

\[
\sigma_e^{(0)} = \left( \frac{1}{2} s_{ij}^{(0)} s_{ij}^{(0)} \right)^{1/2} \quad \text{and} \quad \sigma_e^{(1)} = \frac{3}{2} \frac{s_{ij}^{(0)} s_{ij}^{(1)}}{\sigma_e^{(0)}}.
\]

(4.3)

Substituting (4.1) and (4.2) into the constitutive equation (3.1) we find that

\[
\frac{\epsilon(r, \theta)}{\epsilon_0} = r^{s} \epsilon^{(0)}(\theta) + r^{s+1} \epsilon^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.4)

where

\[
\epsilon^{(0)}_{ij} = \frac{3}{2} \sigma^{(0)} s_{ij}^{(0)},
\]

(4.5)

\[
\epsilon^{(1)}_{ij} = \frac{3}{2} \sigma^{(0)} s_{ij}^{(1)} + \frac{3}{2} (n - 1) \frac{s_{ij}^{(0)} s_{ij}^{(1)}}{\sigma_e^{(0)}}.
\]

(4.6)

and

\[
\epsilon_{ij}^{(0)} = \frac{1 + \nu}{\alpha} s_{ij}^{(0)} + \frac{1 - 2\nu}{2\alpha} \sigma_{ik}^{(0)} \delta_{ij},
\]

(4.7)

By means of a \( J \)-integral argument (Hutchinson, 1968; Rice and Rosengren, 1968) we find that, if the solution is indeed separable to leading order in \( r \) and \( \theta \), then the leading order stress exponent \( s \) must be equal to \(-1/(n + 1)\). The order of magnitude of the second and third terms on the right hand side of the strain expansion (4.4) depends on the value of the second stress exponent \( t \). Restrictions on \( t \) can be determined by substituting (4.4) in the compatibility equation (3.8) and evaluating the dominance of each term as \( r \to 0 \). It is found that \( t \leq (n - 2)/(n + 1) \) (Sharma and Aravas, 1991). The strain and displacement expansion corresponding to the stress expansion (4.1) are

\[
\frac{\epsilon(r, \theta)}{\epsilon_0} = r^{s} \epsilon^{(0)}(\theta) + r^{s+1} \epsilon^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.8)

\[
\frac{\epsilon(r, \theta)}{\epsilon_0} = r^{s} \epsilon^{(0)}(\theta) + r^{s+1} \epsilon^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.9)

when \( t < (n - 2)/(n + 1) \), whereas

\[
\frac{\epsilon(r, \theta)}{\epsilon_0} = r^{s} \epsilon^{(0)}(\theta) + r^{s+1} \epsilon^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.10)

\[
\frac{\epsilon(r, \theta)}{\epsilon_0} = r^{s} \epsilon^{(0)}(\theta) + r^{s+1} \epsilon^{(1)}(\theta) + \cdots \text{ as } r \to 0,
\]

(4.11)

when \( t = (n - 2)/(n + 1) \).

Substituting the expansions (4.1) and (4.8)–(4.11) into the governing equations (3.3)–(3.7) and collecting terms having like powers of \( r \) we obtain the following hierarchy of problems.

For the leading problem we have

\[
(s + 1) \sigma_{r}^{(0)} - \sigma_{\theta}^{(0)} + \frac{d}{d\theta} \sigma_{\theta}^{(0)} = 0,
\]

(4.12)

\[
\frac{d}{d\theta} \sigma_{\theta}^{(0)} + (s + 2) \sigma_{\theta}^{(0)} = 0,
\]

(4.13)
(sn + 1)u_r^{(0)} - \frac{1}{2}\sigma_c^{(0)r-1}s_r^{(0)} = 0,
\text{(4.14)}
\frac{du_r^{(0)}}{d\theta} = \frac{1}{3}\sigma_c^{(0)r-1}s_{\theta\theta} = 0,
\text{(4.15)}
\frac{1}{2} \left( \frac{d u_r^{(0)}}{d \theta} + sn u_\theta^{(0)} \right) - \frac{3}{2}\sigma_c^{(0)r-1}\sigma_{\theta\theta} = 0,
\text{(4.16)}
\text{where } s = -1/(n+1).

To next order the problem is given by
\begin{align*}
(t+1)\sigma_r^{(1)} - \sigma_{\theta\theta}^{(1)} + \frac{d \sigma_{\theta\theta}^{(1)}}{d \theta} &= 0,
\text{(4.17)}
\frac{d \sigma_{\theta\theta}^{(1)}}{d \theta} + (t+2)\sigma_{\theta\theta}^{(1)} &= 0,
\text{(4.18)}
\left[ s(n-1) + t+1 \right] u_r^{(1)}
- \frac{1}{2}\sigma_c^{(0)r-1}\left( s_r^{(0)} + \frac{1}{2}(n-1)\frac{s_{kl}\sigma_{kl}^{(1)}}{\sigma_c^{(0)}}s_{rr}^{(0)} \right) &= 0,
\text{(4.19)}
\frac{1}{2} \frac{du_r^{(1)}}{d\theta} + \left[ s(n-1) + t \right] u_\theta^{(1)}
- \frac{1}{2}\sigma_c^{(0)r-1}\left( \sigma_{\theta\theta}^{(1)} + \frac{1}{2}(n-1)\frac{s_{kl}\sigma_{kl}^{(1)}}{\sigma_c^{(0)}}\sigma_{\theta\theta}^{(0)} \right) &= 0,
\text{(4.20)}
\end{align*}

when \( t < (n-2)/(n+1) \). When \( t = (n-2)/(n+1) \), the corresponding second-order problem becomes
\begin{align*}
(t+1)\sigma_r^{(1)} - \sigma_{\theta\theta}^{(1)} + \frac{d \sigma_{\theta\theta}^{(1)}}{d \theta} &= 0,
\text{(4.22)}
\frac{d \sigma_{\theta\theta}^{(1)}}{d \theta} + (t+2)\sigma_{\theta\theta}^{(1)} &= 0,
\text{(4.23)}
\left[ s(n-1) + t \right] u_r^{(1)}
- \frac{1}{2}\sigma_c^{(0)r-1}\left( s_r^{(0)} + \frac{1}{2}(n-1)\frac{s_{kl}\sigma_{kl}^{(1)}}{\sigma_c^{(0)}}s_{rr}^{(0)} \right) &= 0,
\text{(4.24)}
\end{align*}

Equations (4.12)-(4.26) together with the appropriate asymptotic boundary conditions are used to determine the first two terms in the near-tip expansion of the solution.

It is interesting to note that, when \( t = (n-2)/(n+1) \), the magnitude of \( \sigma^{(1)} \) and \( u^{(1)} \) is completely determined by the magnitude of the leading order solution that appears on the right hand side of (4.24)-(4.26), i.e. the magnitude of the second terms in the stress, strain, and displacement expansions is determined by the J-integral.

Before we proceed to the solution of the above problems, for the purposes of clarity, we normalize the stress, strain and displacements as follows. For \( t < (n-2)/(n+1) \) we write
\begin{align*}
\frac{\sigma(r, \theta)}{\sigma_0} &= \left( \frac{J}{\alpha \epsilon_0 \rho_0 I_n r} \right)^{1/(n+1)} \tilde{\sigma}^{(0)}(\theta)
+ Q \left( \frac{r}{J/\alpha_0} \right)^{1/(n+1)} \tilde{\sigma}^{(1)}(\theta) + \ldots,
\text{(4.27)}
\frac{\epsilon(r, \theta)}{\epsilon_0} &= \left( \frac{J}{\alpha \epsilon_0 \rho_0 I_n r} \right)^{n/(n+1)} \tilde{\epsilon}^{(0)}(\theta)
+ Q \left( \frac{J}{\alpha \epsilon_0 \rho_0 I_n r} \right)^{(n-1)/(n+1)} \times r^{s(n-1)+\tilde{\epsilon}^{(1)}(\theta)} + \ldots,
\text{(4.28)}
\frac{u(r, \theta)}{\alpha \epsilon_0} &= \left( \frac{J}{\alpha \epsilon_0 \rho_0 I_n r} \right)^{(n-1)/(n+1)} \tilde{u}^{(0)}(\theta)
+ Q \left( \frac{J}{\alpha \epsilon_0 \rho_0 I_n r} \right)^{(n-1)/(n+1)} \times r^{s(n-1)+\tilde{u}^{(1)}(\theta)} + \ldots,
\text{(4.29)}
\end{align*}
as \( r \to 0 \), where \( J \) is Rice’s (1968) \( J \)-integral, \( Q \) is a dimensionless constant that controls the magnitude of the second stress term,

\[
\tilde{\varepsilon}_{ij}^{(0)} = \frac{1}{2} \tilde{\sigma}_{e}^{(0)n-1} \tilde{\sigma}_{ij}^{(0)},
\]

(4.30)

\[
\tilde{\varepsilon}_{ij}^{(1)} = \frac{1}{2} \tilde{\sigma}_{e}^{(0)n-1} \left( \tilde{\sigma}_{ij}^{(1)} + \frac{1}{2} (n-1) \frac{\tilde{\sigma}_{k}^{(0)} \tilde{\sigma}_{l}^{(0)}}{(\tilde{\sigma}_{e}^{(0)})^2} \tilde{\sigma}_{ij}^{(0)} \right),
\]

(4.31)

and

\[
n_1 = \cos \theta, \quad n_2 = \sin \theta.
\]

(4.32)

Note that in (4.31) the components \( \tilde{\sigma}_{ij}^{(0)} \) and \( \tilde{\sigma}_{ij}^{(0)} \) are understood to be Cartesian (versus polar). Finally, for \( t = (n-2)/(n+1) \), we write

\[
\frac{\sigma(r, \theta)}{\sigma_0} = \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{1/(n+1)} \tilde{\sigma}_{ij}^{(0)}(\theta)
\]

\[
+ \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{(2-n)/(n+1)} \tilde{\sigma}_{ij}^{(1)}(\theta) + \cdots,
\]

(4.33)

\[
\frac{\epsilon(r, \theta)}{\alpha \epsilon_0} = \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{n/(n+1)} \tilde{\sigma}_{ij}^{(0)}(\theta)
\]

\[
+ \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{1/(n+1)} \tilde{\sigma}_{ij}^{(1)}(\theta) + \cdots,
\]

(4.34)

\[
\frac{u(r, \theta)}{\alpha \epsilon_0} = \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{n/(n+1)} r^{1/(n+1)} \tilde{\sigma}_{ij}^{(0)}(\theta)
\]

\[
+ \left( \frac{J}{\alpha \epsilon_0 \sigma_0 l \rho} \right)^{1/(n+1)} r^{n/(n+1)} \tilde{\sigma}_{ij}^{(1)}(\theta) + \cdots,
\]

(4.35)

as \( r \to 0 \), where now

\[
\tilde{\varepsilon}_{ij}^{(0)} = \tilde{\sigma}_{e}^{(0)n-1} \left[ \tilde{\sigma}_{ij}^{(1)} + \frac{1}{2} (n-1) \frac{\tilde{\sigma}_{k}^{(0)} \tilde{\sigma}_{l}^{(0)}}{(\tilde{\sigma}_{e}^{(0)})^2} \tilde{\sigma}_{ij}^{(0)} \right]
\]

\[
+ \frac{1 + \nu}{\alpha} \tilde{\sigma}_{ij}^{(0)} + \frac{1 - 2\nu}{3\alpha} \tilde{\sigma}_{k}^{(0)} \tilde{\sigma}_{ij}^{(0)}.
\]

(4.36)

### Two-term asymptotic solution

The plane stress condition requires that

\[
\tilde{\sigma}_{ij}^{(0)} = \tilde{\sigma}_{ij}^{(1)} = 0.
\]

(4.37)

Referring to Fig. 1 the boundary conditions for the asymptotic problem are

\[
u_0(r, 0) = 0, \quad \nu_0(r, \pi) = 0, \quad \text{as } r \to 0.
\]

(4.38)

\[
u_0(r, \pi) = 0, \quad \sigma_{00}(r, \pi) = 0, \quad \text{as } r \to 0.
\]

(4.39)

The leading order solution \( (\tilde{\sigma}^{(0)}, \tilde{\sigma}^{(0)} \tilde{u}^{(0)}) \) is determined from the solution of the homogeneous non-linear differential equations (4.12)–(4.16) and the homogeneous boundary conditions (4.38) and (4.39). The system of equations (4.12)–(4.16) is numerically solved for different values of the hardening exponent using a Galerkin-finite-element technique. Two-node isoparametric elements with two Gauss integration points are used in the computations. Five degrees of freedom per node are used and the interpolation of the unknowns \( (\tilde{u}_r^{(0)}, \tilde{u}_r^{(0)}, \tilde{u}_r^{(0)}, \tilde{u}_{00}^{(0)}, \tilde{u}_{00}^{(0)} \) within the element is defined by a first-order polynomial in terms of their nodal values. The discretized form of the governing equations (4.12)–(4.16) consist of a set of non-linear homogeneous algebraic equations which are solved using Newton’s method. A total of 1440 elements is used in the computations. The solution is determined to within a multiplicative constant, which is chosen so that \( \tilde{\sigma}_{ij}^{(0)} = 1 \) and \( \tilde{\sigma}_{00}(\pi) < 0 \).

Once the leading order solution is found, Eqs. (4.17)–(4.26) can be used to determine the next-order terms in the expansion of the solution. In order to determine second-order term it is assumed first that \( t < (n-2)/(n+1) \). In this case the exponent \( t, \tilde{\sigma}^{(1)} \) and \( \tilde{u}^{(1)} \) are determined from the linear eigenvalue problem defined by Eqs. (4.17)–(4.21) and the boundary conditions (4.38) and (4.39). The system of equations (4.17)–(4.21) is solved numerically for different values of the hardening exponent using a Galerkin-finite-element technique similar to that used for the solution of the leading-order problem. The discretized form of the governing equations (4.17)–(4.21) consists of a set of homogeneous linear algebraic equations, the coefficients of which depend on the value of \( t \). For a non-trivial solution to exist, the determinant of the matrix of coefficients must
vanish, and this condition defines the eigenvalue \( t \).
In general, there will be more than one real values of \( t \) greater than \(-1/(n + 1)\) for which the aforementioned determinant vanishes and of all such values the smallest one must be chosen. If the value of \( t \) is indeed less than \((n - 2)/(n + 1)\) then the correct solution has been found and we proceed to determine \( \tilde{\varphi}^{(1)} \) and \( \tilde{u}^{(1)} \).
If the value of \( t \) obtained from the eigenvalue problem (4.17)–(4.21) is greater than \((n - 2)/(n + 1)\) then the original assumption that \( t < (n - 2)/(n + 1) \) is incorrect and the actual value of \( t \) is \( t = (n - 2)/(n + 1) \). In this case \( \tilde{\varphi}^{(1)} \) and \( \tilde{u}^{(1)} \) are determined from the solution of the non-homogeneous linear system given by Eqs. (4.22)–(4.26) which is also solved using a Galerkin-finite-element technique.

Figure 2 shows the angular variations of the

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**Fig. 2.** Angular variation of the leading order in-plane stress and displacement components for \( n = 3 \) and \( n = 10 \).
leading order stress and displacement components for hardening exponents \( n = 3 \) and \( n = 10 \). The variation of \( I_n \) with \( n \) is shown in Fig. 3.

The leading-order and the next-order solutions thus obtained have some interesting features. Depending on the value of the hardening exponent \( n \), the leading order solutions indicate that the condition \( \sigma_{\theta \theta}(\theta = \pi) < 0 \), which is needed to keep the crack closed as \( r \to 0 \), requires that

\[
\sigma_{\theta \theta}(\theta = 0) < 0 \quad \text{for} \quad 1 < n < 5.7 \tag{4.40}
\]

and

\[
\sigma_{\theta \theta}(\theta = 0) > 0 \quad \text{for} \quad n > 5.7. \tag{4.41}
\]

Put in other words, a necessary condition for the crack to remain closed near its tip, where plastic yielding occurs and the first term on the right hand side of (4.27) or (4.33) is the dominant term, is that \( \sigma_{\theta \theta} \) on \( \theta = 0 \) should take corresponding values specified by Eqs. (4.40) and (4.41). It is interesting to note that for the crack to be closed, the elastic solution requires that \( K_{II} < 0 \) or, equivalently, \( \sigma_{\theta \theta}(\theta = 0) < 0 \) as \( r \to 0 \). Some interesting implications of these observations are discussed in Section 5.

The results of the calculations for the next-order term show that

\[
t = \frac{n-2}{n+1} \quad \text{for} \quad 1 < n \leq 2.1
\]

and

\[
t < \frac{n-2}{n+1} \quad \text{for all other} \quad n. \tag{4.43}
\]

The values of the second stress exponent \( t \) are listed in Table 1 for integer values of the hardening exponent \( n \) in the region \( 2 \leq n \leq 20 \). It should be noted that for \( 5 \leq n \leq 20 \), a range typical to ductile metals, the second stress exponent \( t \) is greater than 0.1. This is in contrast to the plane strain results where \( t \) was found to be zero for all \( n \) (Sharma and Aravas, 1991). This suggests that the region of dominance of the leading term in the stress expansion would, in general, be larger in plane stress than in plane strain.

Figure 4 shows the angular variation of \( \sigma^{(1)}(\theta) \) and \( \bar{u}^{(1)}(\theta) \) for two values of the hardening exponents \( n = 3 \) and \( n = 10 \).

In the next section we compare the asymptotic solutions developed here with the results of detailed finite element calculations.

5. Finite element analysis

In order to verify the asymptotic results developed in the previous section, we carry out detailed
finite element analysis of a Griffith crack lying along the interface of an elastoplastic medium and a rigid substrate (Fig. 5), subjected to remote shear loading. The constitutive behavior of the deformable medium is described by the $J_2$ deformation theory for a Ramberg-Osgood uniaxial relation presented in Section 3. Any contact between the crack face and the rigid substrate is assumed to be frictionless.

The finite element model is constructed using 9-node Lagrangian isoparametric elements with
The ABAQUS general purpose finite element program (Hibbitt, 1984) is used for computations. It should be mentioned here that the 9-node element is not included in the ABAQUS "element library". However, the user has the option of introducing his/her own element using the user interface provided by the code. The constitutive behavior is part of the element definition and the deformation plasticity model discussed in Section 3 were implemented through the user interface.

The material properties used in the calculations are $E/o_o = 300$, $v = 0.2$, and $a = 0.1$. The finite element analysis was carried out for two different values of the hardening exponent namely, $n = 3$ and $n = 10$.

In both cases, the applied shear traction $\sigma_{xy}^\infty$ is gradually increased from zero to $\sigma_0/\sqrt{3}$ where the calculations are terminated.

**Results and discussion.** Before proceeding to discuss the results obtained we reconsider Eqs. (4.40) and (4.41) in order to facilitate certain explanations regarding crack closure. Referring to the coordinate system centered at the left crack tip as shown in Fig. 5 and noting that $\theta = 0$ on the crack face, the condition $\sigma_{\theta\theta}(\theta = 0) < 0$, needed to keep the left crack tip closed as $r \to 0$, is equivalent to

$$\sigma_{r\theta}(\theta = \pi) > 0 \quad \text{as } r \to 0 \quad \text{for } 1 \leq n < 5.7 \quad (5.1)$$

and

$$\sigma_{r\theta}(\theta = \pi) < 0 \quad \text{as } r \to 0 \quad \text{for } n > 5.7 \quad (5.2)$$

where $\theta = \pi$ refers to the bonded interface ahead of the left crack tip (see Fig. 5).

For $n = 3$, it is found that the right crack tip opens while the left crack tip remains closed. The contact zone $c_1$ measured from the left crack tip remains constant during the deformation process and is equal to that predicted by the elastic solution namely $c_1 = 0.6a$ (Comminou, 1978). Further, the shear stress ahead of the left crack tip is positive so that (5.1) is satisfied. The contact zone at the right crack tip, if any, could not be captured by the mesh used in the finite element calculations and it appears that for most practical cases the right crack tip can be treated as an open tip.

The finite elements results presented in the following are at the final load level $\sigma_{xy}^\infty = \sigma_0/\sqrt{3}$. Figures 6 and 7 show the computed radial variation of the normalized hydrostatic and Mises equivalent stress respectively along the radial line $\theta = 142.5^\circ$ ahead of the left crack tip (see Fig. 5). The predictions of the asymptotic solution are also plotted in Figs. 6 and 7. In these and all the subsequent figures the finite element results are indicated by open circles, curve I indicates the leading order term in the asymptotic solution, whereas curve II is the sum of the first two terms on the right hand side of (4.27).
Fig. 7. Radial variation of the normalized mises equivalent stress $\sigma_e$ along $\theta = 142.5^\circ$ for $n = 3$. The open circles indicate the results of finite element solution. Curve I represents the leading order term in the asymptotic solution, whereas curve II is the sum of the first two terms on the right hand side of (4.27).

This local crack opening can be easily explained in terms of the asymptotic solution developed in the previous section. According to Eq. (5.1), for values of $n > 5.7$, the shear stress $\sigma_{\theta \phi}$ ahead of the left crack tip should be negative for the crack to be closed. However, if we consider the entire deformation process, from elastic to plastic, ahead of that tip, we find that the shear stress $\sigma_{\theta \phi} (\theta = \pi)$, which is positive when the crack is closed during elastic deformation, remains so during the plastic deformation as well. Thus, for $n = 10$, the necessary condition for the left crack tip to be closed when yielding occurs at the tip is never realised and the left tip opens locally.

At larger distances from the tip, the crack remains closed and higher order terms in the stress expansion (4.33) are needed to explain the crack closure (Aravas and Sharma, 1991). It is interesting to note, however, that, for $n = 10$, the magnitude of the second terms in the stress expansion is determined by the $J$-integral and that the second stress exponent takes the value $t = (n - 2)/(n + 1) = 0.727$; this relatively high value of $t$ makes the contribution of the second term in the stress expansion insignificant as $r \to 0$. We also find that, on the crack face, $\tilde{\sigma}_{\theta \phi}^{(1)}$ always has the same sign as $\tilde{\sigma}_{00}^{(0)}$ and, therefore, the two-term stress expansion (4.33) does not explain the crack closure at the left crack tip. This suggests that either more terms in the asymptotic expansion must be considered in order to explain crack closure, or the second term in the stress expansion is not of the form assumed in (4.1) (i.e. it is not separable in $r$ and $\theta$).

The results of the finite element solution for $n = 10$ are compared in the following with the predictions of the asymptotic solution at the final load level $\sigma_{\phi}^{\infty} = \sigma_0/\sqrt{3}$. The two solutions are compared at the left crack tip over radial distances that are larger than the open portion $d_1$ but still small compared to the size of the contact zone $c_1$. Figures 9 and 10 show the radial variation of the normalized hydrostatic and Mises equivalent stress together with the asymptotic plastic solution along the radial line $\theta = 142.5^\circ$ ahead of the left crack tip. Figure 11 shows the angular variation of the normalized stress components $\sigma_{ij}/\{ \sigma_0 J/ (\alpha \epsilon_0 \sigma_0 I_{r} r) \}^{1/(n+1)}$ and the normalized Mises equivalent stress at a radial distance $r = 10^{-3} a$. 

Next, we consider the results for $n = 10$. Once again, the applied shear traction is gradually increased from zero to a value of $\sigma_{\phi}^{\infty} = \sigma_0/\sqrt{3}$ where the calculations are terminated. It is found that the size of the contact zone at the left crack tip remains constant at all load levels, namely $c_1 = 0.6a$, while the right crack tip is open. It is interesting to note, however, that the left crack tip also opens locally within the contact zone; the size of the open portion is always a small fraction of the contact zone and it increases with increasing load to a value of $d_1 = 7.94 \times 10^{-5} a$ at $\sigma_{\phi}^{\infty} = \sigma_0/\sqrt{3}$. 

The results of the finite element solution for $n = 10$ are compared in the following with the predictions of the asymptotic solution at the final load level $\sigma_{\phi}^{\infty} = \sigma_0/\sqrt{3}$. The two solutions are compared at the left crack tip over radial distances that are larger than the open portion $d_1$ but still small compared to the size of the contact zone $c_1$. Figures 9 and 10 show the radial variation of the normalized hydrostatic and Mises equivalent stress together with the asymptotic plastic solution along the radial line $\theta = 142.5^\circ$ ahead of the left crack tip. Figure 11 shows the angular variation of the normalized stress components $\sigma_{ij}/\{ \sigma_0 J/ (\alpha \epsilon_0 \sigma_0 I_{r} r) \}^{1/(n+1)}$ and the normalized Mises equivalent stress at a radial distance $r = 10^{-3} a$. 

Fig. 7. Radial variation of the normalized mises equivalent stress $\sigma_e$ along $\theta = 142.5^\circ$ for $n = 3$. The open circles indicate the results of finite element solution. Curve I represents the leading order term in the asymptotic solution, whereas curve II is the sum of the first two terms on the right hand side of (4.27).
together with the asymptotic plastic solution. The contribution of the second term in the stress expansion (4.33) is insignificant at \( r = 10^{-3}a \) and is not included in Figs. 9–11. We find that the leading term in stress expansion provides a reasonable approximation to the solution for values of \( r \) in the range \( d_1 < r < 10^{-3}a \).

Summarizing, we mention that the predictions of the asymptotic solution developed in Section 4 agree well with the results of the finite element calculations.

6. The case of two deformable media

We conclude the paper with a brief discussion on the possible application of the solutions developed in Section 4 to the problem of a crack along

![Graphs showing angular variation of normalized stress components for different values of \( \theta \) and \( n = 3 \). The open circles indicate the results of finite element solution.](image)

Fig. 8. Angular variation of the normalized stress components for \( n = 3 \). The open circles indicate the results of finite element solution.
the interface of two deformable elastoplastic media as shown in Fig. 12. Both materials are characterized by the Young's modulus $E$, Poisson's ratio $v$, and the plasticity constants $n$ and $c$. For definiteness, we assume that $n_1 < n_2$, where materials I and II are labeled as shown in Fig. 12.

Referring to the closed crack tip shown in Fig. 12, and assuming the crack surfaces to be in frictionless contact, the boundary conditions for the asymptotic problem are

$$\sigma_{\theta \theta}(\theta = \pm \pi) = 0,$$

$$\sigma_{\theta \theta}(\theta = \pi) = \sigma_{\theta \theta}(\theta = -\pi) \leq 0,$$

$$u_{\theta}(\theta = \pi) = u_{\theta}(\theta = -\pi),$$

as $r \to 0$. The continuity conditions along the interface are

$$[u_i] = 0 \text{ and } [\sigma_{ii}] = [\sigma_{\theta \theta}] = 0,$$

(6.2)

where $[ ]$ denotes jumps of the functions across the interface.

If we assume that the near tip stress expansion is of the form

$$\frac{\sigma(r, \theta)}{\sigma_0} = r^s \phi^{(s)}(\theta) + r^{s+1} \phi^{(s+1)}(\theta) + \cdots$$

(6.3)

as $r \to 0$, in both materials (i.e. in the whole region $-\pi \leq \theta \leq \pi$), then the corresponding strain and displacement expansions are

$$\frac{\epsilon_1(r, \theta)}{\alpha_1 \epsilon_1(0)} = r^{s_1} \epsilon_1^{(s_1)}(\theta) + r^{s_1+1} \epsilon_1^{(s_1+1)}(\theta) + \cdots$$

as $r \to 0$,

$$\frac{u_1(r, \theta)}{\alpha_1 \epsilon_1(0)} = r^{s_1+1} u_1^{(s_1+1)}(\theta) + \cdots$$

(6.4)

as $r \to 0$, in material I, and

$$\frac{\epsilon_{II}(r, \theta)}{\alpha_{II} \epsilon_{II}(0)} = r^{s_{II}} \epsilon_{II}^{(s_{II})}(\theta) + r^{s_{II}+1} \epsilon_{II}^{(s_{II}+1)}(\theta) + \cdots$$

as $r \to 0$,

$$\frac{u_{II}(r, \theta)}{\alpha_{II} \epsilon_{II}(0)} = r^{s_{II}+1} u_{II}^{(s_{II}+1)}(\theta) + \cdots$$

(6.5)

as $r \to 0$, in material II.

Using a $J$-integral argument similar to that used by Hutchinson (1968) and Rice and Rosengren (1968) and taking into account that $n_1 < n_2$ we can readily show that $s = -1/(n_1 + 1)$ (see also Yuli Gao and Zhiwen Lou (1990) and Champion and Atkinson (1990, 1991)).

Substituting the strain expansions (6.4) and (6.6) into the compatibility equation (3.8) and balancing terms, we conclude that (Sharma and Aravas, 1991)

$$t \leq \frac{n_1 - 2}{n_1 + 1},$$

(6.8)

which makes the third term on the right hand side of (6.7) smaller than the first two terms as $r \to 0$.

The displacement continuity condition (6.2a) now becomes

$$\alpha_{II} \epsilon_{II} \left[r^{s_{II}+1} u_{II}^{(s_{II})}(0) + \cdots \right] - \alpha_1 \epsilon_1 \left[r^{s_1+1} u_1^{(s_1)}(0) + \cdots \right] = 0,$$

(6.9)
as $r \to 0$. Taking into account that $n_1 < n_{II}$, we conclude that the displacement continuity condition reduces, to leading order, to

$$u_{II}^{(0)}(0) = 0. \quad (6.10)$$

When $s(n_{II} - 1) + t + 1 < s n_1 + 1$ (i.e. $t < s(n_1 - n_{II} + 1)$) (6.9 yields

$$u_{II}^{(1)}(0) = 0. \quad (6.11)$$

to next order. When $t = s(n_1 - n_{II} + 1)$, (6.9) yields

$$u_{II}^{(1)}(0) = \frac{\alpha_{II}^0}{\alpha_{II}^{II}} u_{II}^{(0)}(0), \quad (6.12)$$

to next order. Finally, the possibility $t > s(n_1 - n_{II} + 1)$ must be ruled out, because, in that case, the second term in (6.9), which is completely defined in terms of the leading-order solution, is larger than any of the subsequent terms and cannot be balanced as $r \to 0$. Combining this restriction on $t$ with (6.8), we conclude that

$$t \leq \min \left( \frac{n_1 - 2}{n_{II} + 1}, \frac{n_{II} - n_1 - 1}{n_{II} + 1} \right). \quad (6.13)$$

Similarly, the displacement boundary condition (6.1b) reduces, to leading order, to

$$u_{II\theta}^{(0)}(\pi) = 0. \quad (6.14)$$

To next order, (6.1b) yields

$$u_{II\theta}^{(1)}(\pi) = 0. \quad (6.15)$$

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![Fig. 10. Radial variation of the normalized mises equivalent stress $\sigma_e$ along $\theta = 142.5^\circ$ for $n = 10$. The open circles indicate the results of finite element solution.](image)
Fig. 11. Angular variation of the normalized stress components for \( n=10 \). The open circles indicate the results of finite element solution.

\[
E_2, \nu_2, \alpha_2, n_2 \quad \text{and} \quad E_1, \nu_1, \alpha_1, n_1
\]

Fig. 12. Crack along the interface of two deformable media.

if \( t < s(n_1 - n_{11} + 1) \), and

\[
\varepsilon_{10}^{(\sigma_0)}(\pi) = \frac{\alpha_{10} \varepsilon_0}{\alpha_{11} \varepsilon_0} \varepsilon_{10}^{(\sigma_0)}(-\pi),
\]

if \( t = s(n_1 - n_{11} + 1) \).

Summarizing, we mention that the boundary
and continuity conditions for material II reduce, to leading order, to

\[ u_{i\text{II}}^{(0)}(0) = 0, \quad u_{i\text{II}}^{(0)}(0) = 0, \quad (6.17) \]

and

\[ u_{i\text{II}}^{(0)}(\pi) = 0, \quad \sigma_{\theta}^{(0)}(\pi) = 0. \quad (6.18) \]

The leading-order boundary conditions for material II (6.17) and (6.18) are identical to those used in the two-term asymptotic solution section (see (4.38) and (4.39)) for the rigid substrate; put in other words, material I in Fig. 12, where \( n_1 < n_{\text{II}} \), acts, to leading order, as rigid substrate in the limit \( r \to 0 \). This shows that the leading order solution \( (\bar{\sigma}^{(0)}, \bar{\mathbf{u}}^{(0)}) \) developed in Section 4 for a rigid substrate is a valid leading order solution for material II in Fig. 12. The second order solution will, of course, be different in this case.

7. Closure

A detailed analysis of the plane stress interface crack problem with frictionless contact zones has been presented in this paper. The substrate was assumed to be rigid and the results of the analysis can be used for the study of crack-like defects along the interface of a ductile metal and a brittle substrate, such as a ceramic.

Any friction forces along the contact zones were ignored in our analyses. Such frictional effects are certainly important and will be addressed in the future. Comninou (1977b) presented an elastic solution for the interface crack with contact zones using Coulomb’s model for the frictional forces. She found that the near tip stresses are less than square-root singular and that the strength of the elastic singularity decreases as the coefficient of friction increases. This implies, in turn, that the elastic energy release rate associated with crack extension vanishes. This seems to be an artifact of the use of Coulomb’s law in a region where singular stresses develop. It appears, therefore, that for further progress towards a more complete understanding of interfacial fracture more realistic friction models for the near tip region are needed.

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