Finite element implementation of gradient plasticity models
Part II: Gradient-dependent evolution equations

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Abstract

The variational formulation and finite element implementation of a class of plasticity models with gradient-dependent yield functions is covered in detail in Part I. In this sequel, attention is focussed upon the finite element formulation and implementation of a different class of plasticity models wherein spatial gradients of one or more internal variables enter the evolution equations for the state variables. Finite element solutions are obtained for the problems of localization of plastic flow in plane strain tension and of a mode-1 plane strain crack. The pathological dependence of the finite element solution on the size of the elements in local plasticity models disappears when the gradient-type model is used. © 1998 Elsevier Science S.A. All rights reserved.

1. Introduction

A general review of research on non-local continuum models in the context of plastic deformation is presented in the Introduction of Part I of our work (this issue), wherein the formulation of a general class of nonlocal rate-independent plasticity models involving gradient-dependent yield functions is described in detail. In this sequel, we focus our attention upon a different family of rate-independent gradient-type plasticity models in which spatial gradients of the plastic multiplier and state variables enter the evolution equations for the state variables. As an example to this class of models, we present a gradient-dependent version of Gurson’s [4] plasticity model for porous metals, in which the classical evolution equation for the porosity is now replaced by a gradient-dependent evolution equation. The modified equation involves spatial gradients of porosity and is therefore of a spatio-temporal nature as opposed to an ordinary rate equation in the classical model. Emphasis is primarily placed on the Galerkin finite element formulation and implementation of the gradient-dependent model along with related issues such as the interpretation of ‘plastic loading’ in the discretized problem.

Among related works involving the numerical treatment of non-local models in the context of porous ductile materials, we mention the recent papers by Tvergaard and Needleman [6,7], which use an elastic-viscoplastic version of Gurson’s model and describe the evolution of the porosity through an integral equation as presented in [5].

The organization of this paper is as follows. Section 2 outlines the general form of plastic constitutive equations with gradient-dependent evolution equations for the state variables. A review of the standard Gurson model is also presented, which is followed by the presentation of the gradient-dependent Gurson’s model. The
full boundary value problem for an elastic-plastic body with gradient-dependent evolution equations is outlined in Section 3. The description of the mixed finite element formulation used in the solution of the problem is given in Section 4. The specific application to the Gurson’s model together with issues such as the numerical integration of the constitutive equations, the conditions for ‘plastic loading’, and the linearization of the finite element equations are described in detail in Section 5. Finally, Section 6 presents finite element solutions for the problems of localization of plastic flow in plane strain tension and of a mode-I plane strain crack. It is found that the pathological dependence of the numerical solution on the size of the finite elements used in local plasticity models disappears when the gradient-type model is used.

The tensor notation used in this paper is the same as that of Part I.

2. Plastic constitutive equations of the gradient-type

Classical rate-independent plasticity models are briefly reviewed in Part I wherein the formulation of a general class of nonlocal plasticity models involving gradient-dependent yield functions is described in detail. In this sequel, attention is focussed upon a different family of gradient-type plasticity models in which the gradient terms enter the evolution equations for the state variables.

In particular, we consider plasticity models of the form

$$\dot{\Phi}(\sigma, s_1, \ldots, s_n) = 0, \quad \dot{\lambda} = 0, \quad \dot{\lambda} = 0, \quad \dot{\Phi} = 0,$$  

$$D^p = \hat{\lambda}N(\sigma, s_1, \ldots, s_n),$$  

$$s_\alpha = \hat{s}_\alpha(\sigma, s_1, \ldots, s_n, \nabla s_\alpha, \nabla s_\alpha, \lambda, \nabla \lambda), \quad \alpha = 1, \ldots, n,$$  

where $\Phi$ is the yield function in stress space, $\sigma$ is the Cauchy stress tensor, $s_\alpha (\alpha = 1, \ldots, n)$ is a collection of scalar state variables, $\lambda$ is a non-negative plastic multiplier, $N$ and $\hat{s}_\alpha$ are isotropic functions of their arguments. Since we deal with rate-independent materials, the functions $\hat{s}_\alpha$ are homogeneous of degree one in $\lambda$.

The presence of gradients in the evolution equations for the state variables requires additional boundary conditions on the boundary of the plastic zone. The general form of these boundary conditions is

$$\dot{\lambda} = 0 \quad \text{on } S_p,$$  

$$s_\alpha = \vec{s}_\alpha, \quad \alpha = 1, \ldots, n, \quad \text{on } S_p,$$  

$$\frac{\partial s_\alpha}{\partial n} = \vec{s}_\alpha, \quad \alpha = 1, \ldots, n, \quad \text{on } S_p,$$  

where $\vec{s}_\alpha$ and $\vec{s}_\alpha$ are known functions over $S_p$ and $S_p$, respectively, $\vec{n}$ is the outward unit normal to $S_p$, and $S_p \cup S_p - S_p$, where, as defined in Part I, $S_p$ is the boundary of the plastic zone $V_p$, $S_p$ is the elastic-plastic interface, and $S_p = S_p \cap S$, where $S$ is the outer surface of the body.

2.1. Example: Gurson’s plasticity model

This section presents a brief review of the Gurson [4] model for porous metals, and a suggested modification to it of the gradient-type.

2.1.1. Review of the Gurson model

The yield function depends on the linear invariant of $\sigma$ and the quadratic invariant of $\sigma'$. The model involves two state variables ($\alpha = 2$): (i) the equivalent microscopic plastic strain in the matrix material $\varepsilon_p^m$, and (ii) the ‘porosity’ $f$, which is defined as the volume fraction of the voids. The yield condition is of the form

$$\Phi(p, q, \varepsilon_p^m, f) = \left[ \frac{q}{\sigma_m(\varepsilon_p^m)} \right]^2 + 2 f \cosh\left[ \frac{3p}{2\sigma_m(\varepsilon_p^m)} \right] - (1 + f^2) = 0,$$  

where $q = \sqrt{(3/2)\sigma_0'\sigma_0'}$ is the von Mises equivalent stress, $\sigma'$ being the deviatoric part of $\sigma$, $p = -\sigma_{kk}/3$ is the hydrostatic stress (positive in compression), and $\sigma_m$ is the flow stress of the matrix material.

The yield function is used as a plastic potential ($N = \partial \Phi/\partial \sigma$), so that
The evolution equation for the first state variable $\dot{\varepsilon}^p$ is based on the requirement that the macroscopic 'plastic work' $\sigma : D^p$ equals the microscopic one $(1 - f)\sigma_m \dot{\varepsilon}^p$, so that

$$\dot{s}_1 = \dot{\varepsilon}^p = \frac{\sigma : D^p}{(1 - f)\sigma_m} = \frac{\sigma : N}{(1 - f)\sigma_m}. \tag{8}$$

As the porous metal deforms plastically, its porosity can change due to the growth (or closure) of the existing voids, and due to the nucleation of new voids, i.e.

$$\dot{s}_2 = \dot{f} = \dot{f}_{gr} + \dot{f}_{nuc}. \tag{9}$$

Assuming that the matrix material is plastically incompressible and ignoring the elastic contribution to the void-volume change, one can readily show that

$$\dot{f}_{gr} = (1 - f)D^p_{kk} = \dot{\lambda}(1 - f)N_{kk}. \tag{10}$$

We consider plastic strain-controlled nucleation such as

$$\dot{f}_{nuc} = \mathcal{A} \dot{\varepsilon}^p, \tag{11}$$

where, as suggested by Chu and Needleman [3], the parameter $\mathcal{A}$ is chosen so that the nucleation strain follows a normal distribution with mean value $\varepsilon_n$ and standard deviation $s_N$:

$$\mathcal{A} = \frac{f_N}{s_N \sqrt{2\pi}} \exp \left[-\frac{1}{2} \left(\frac{\dot{\varepsilon}^p - \varepsilon_n}{s_N}\right)^2\right]. \tag{12}$$

2.1.2. A gradient-type modification to the evolution equation of porosity

In this section, we discuss different forms of the evolution equation for porosity which involve additional terms that account for such effects as void diffusion, interaction, and coalescence. The derivation of these evolution equations is discussed in the following.

We consider an arbitrary volume $\Omega$ of the porous elastic-plastic continuum. The amount of 'void space' in $\Omega$ can change due to the growth (or closure) of the existing voids, the nucleation of new voids, and the flux of voids that enter $\Omega$ by crossing its boundary $\partial \Omega$. Therefore, we can write

$$\int_{\Omega} \dot{f} \, dV = \int_{\Omega} (\dot{f}_{gr} + \dot{f}_{nuc}) \, dV - \int_{\partial \Omega} j \cdot \hat{n} \, dS, \tag{13}$$

where $\hat{n}$ is the outward normal to $\partial \Omega$, and $j$ is the void flux vector. The negative sign on the right-hand side of the last equation is needed because $\int_{\partial \Omega} j \cdot \hat{n} \, dS$ is the outward void flux. Using Green's theorem and taking into account that the volume $\Omega$ is arbitrary, we conclude readily that

$$\dot{f} = \dot{f}_{gr} + \dot{f}_{nuc} - \nabla \cdot j. \tag{14}$$

The equation that describes $j$ depends on the physical mechanism that causes the flux of voids. Here, we consider two such mechanisms: (i) diffusion, and (ii) void interaction and coalescence in a plastically deforming metal. In the case of diffusion, the simplest form for $j$ would result if we assume that the flux is proportional to the porosity gradient:

$$j = -c \nabla f, \tag{15}$$

where $c$ is a constant diffusion coefficient ($c > 0$) with dimensions (length)$^2$/time. The negative sign in the above equation is due to the fact that voids will move in a manner which will decrease the porosity gradients (see Fig. 1(a)). In this case, Eq. (15) becomes

$$\dot{f} = \dot{\lambda}(1 - f)N_{kk} + \mathcal{A} \dot{\varepsilon}^p + c \nabla^2 f. \tag{16}$$
Next, we consider the case of interacting voids in a plastically deforming matrix. Here, one can argue that, as the matrix deforms, larger voids attract smaller voids and, eventually, coalesce with them (see Fig. 1(b)); i.e., the void flux now is in the opposite direction of that in diffusion. A simple expression for $j$ that has this feature is

$$j = \dot{\lambda} \frac{\ell^2}{\sigma_0} \nabla \cdot \nabla f,$$

where $\ell$ is a material length scale, and $\sigma_0$ a reference stress. In the above equation, the void flux is taken proportional to the 'amount of the rate of plastic straining', as measured by $\dot{\lambda}$, which controls the magnitude of $D^p$. We also note that $j$ is now non-zero only when plastic flow occurs ($\dot{\lambda} \neq 0$). In this case, Eq. (15) becomes

$$\dot{f} = \dot{\lambda}(1-f)N_{kk} + \dot{\lambda}\dot{\varepsilon}^p - \frac{\ell^2}{\sigma_0} \nabla \cdot (\dot{\lambda} \nabla f).$$

We conclude this section by emphasizing the very different effects that the gradient terms in Eqs. (17) and (19) are expected to have. In the case of diffusion, the term $-\epsilon \nabla^2 f$ in (17) will tend to eliminate any existing porosity gradients; whereas the gradient terms in (19) will have a 'destabilizing' effect in the solution, in that they tend to intensify any existing porosity gradients during plastic flow.

The presence of the gradient terms in the evolution equation of porosity necessitates additional boundary conditions for $f$ on the boundary of the plastic zone. On the elastic-plastic interface $S^p$, the value of porosity is specified. In particular, if a point $A$ on $S^p$ yields for the first time, the boundary condition at that point is $f = f_0$, $f_0$ being the initial porosity in the material; on the other hand, if point $A$ has yielded previously, the corresponding value of the porosity equals the value of $f$ when the point was last actively yielding. We also assume that no voids can 'enter' the continuum, so that $\mathbf{j} \cdot \mathbf{n} = 0$ on the part of the boundary of the plastic zone $S^p$ that belongs to the outer surface of the body. This condition is equivalent to

![Fig. 1. Schematic representation of flux of voids.](image-url)
3. The elastic-plastic boundary value problem

We consider an elastic-plastic continuum, which in its undeformed state occupies a volume \( V_0 \) in space. Let \( V \) be the corresponding volume of the continuum in its deformed state. We formulate the general problem for the elastic-plastic material of the

\[
\nabla \cdot \sigma + b = 0,
\]

\[
D = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T],
\]

\[
D = D^e + D^p,
\]

\[
\ddot{\sigma} = C^e : \dot{D}^e,
\]

\[
D^p = \dot{\lambda} N(\sigma, s), \quad N = \frac{\partial \Phi}{\partial \sigma},
\]

\[
\dot{\Phi} (\sigma, s_1, \ldots, s_n) \leq 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} \Phi = 0,
\]

\[
\dot{s}_{\alpha} = \ddot{s}_{\alpha}(\sigma, s_1, \ldots, s_n, \nabla s_{\alpha}, \nabla^2 s_{\alpha}, \dot{\lambda}, \nabla \dot{\lambda}), \quad \alpha = 1, \ldots, n,
\]

in \( V \), where the notation of Part I is used, and all gradients are evaluated in the current deformed configuration. The corresponding boundary conditions are

\[
u = \bar{u} \quad \text{on } S_u,
\]

\[
n \cdot \sigma = \bar{t} \quad \text{on } S_t,
\]

\[
\dot{\lambda} = 0 \quad \text{on } S^p,
\]

\[
s_{\alpha} = \bar{s}_{\alpha}, \quad \alpha = 1, \ldots, n \quad \text{on } S^p,
\]

\[
\frac{\partial \bar{s}_{\alpha}}{\partial n} = \bar{s}_{\alpha}, \quad \alpha = 1, \ldots, n \quad \text{on } S^p,
\]

where \( \bar{u} \) denotes the prescribed displacements over \( S_u \), \( \bar{t} \) the prescribed tractions over \( S_t \) which has an outward unit normal \( n \), and \( S_u \cup S_t = S \), \( S \) being the boundary of \( V \).

4. Variational formulation

A variational formulation of the boundary value problem listed in Section 3 is developed in the following. In order to derive the discrete governing equations, we first integrate the evolution equations for the state variables over the time interval \([t_n, t_{n+1} = t_n + \Delta t] \) with a backward-Euler scheme:

\[
\Delta s_{\alpha} = \ddot{s}_{\alpha}(\sigma_{n+1}, s_1, \ldots, s_n, (\nabla s_{\alpha})_{n+1}, (\nabla^2 s_{\alpha})_{n+1}, \Delta \lambda, (\Delta \lambda)),
\]

where we took into account that the functions \( \ddot{s}_{\alpha} \) are homogeneous of degree one in \( \lambda \).

We start by expressing the equilibrium equations (21), the traction boundary conditions (29), the yield condition (26a), the evolution equations for the state variables (33), and the boundary conditions (32), in the following variational statement: Find

(1) \( u(x, t) \in H^2(V) \) satisfying \( u|_{S_u} = \bar{u} \),

(2) \( \lambda(x, t) \in H^1(V) \) satisfying \( \lambda|_{S^p} = 0 \), and

(3) \( s_{\alpha}(x, t) \in H^2(V), \quad \alpha = 1, \ldots, n \) satisfying \( s_{\alpha}|_{S^p} = \bar{s}_{\alpha} \),

such that for all \( v \in L^2(V) \) satisfying \( v|_{S_u} = 0 \), for all \( \lambda \in L^2(V) \), for all \( q^*_\alpha \in L^2(V) \), and for all \( a^*_\alpha \in L^2(V) \) satisfying \( a^*_\alpha|_{S^p} = 0 \),
\[ A(u, \lambda, s_1, \ldots, s_n, v^*, \lambda^*, q_1^*, \ldots, q_n^*, a_1^*, \ldots, a_n^*) = \int_V (\sigma_{ij,j} + b_i) v_i^* \, dV + \int_S (\lambda_{ij}) v_i^* \, dS \]
\[ + \int_{\partial V} \left[ \sum_{\alpha=1}^n \left[ \Delta s_\alpha - s_\alpha (\sigma, s_1, \ldots, s_n, V_s, \nabla s_\alpha, \Delta \lambda, V(\Delta \lambda)) \right] q_\alpha^* \right] \, dV \]
\[ + \int_{\partial V} \left[ \sum_{\alpha=1}^n \left( s_\alpha \bar{p}_{ij} - \bar{s}_{\alpha} \right) a_\alpha^* \right] \, dS = 0, \tag{34} \]

where \( \sigma = \sigma(u, \lambda, s_\alpha) \), and a comma denotes partial differentiation with respect to position, i.e. \( A_{,i} = \partial A / \partial x_i \).

5. Application: The Gurson model

We consider the case where the evolution equation for porosity is modified to account for void interaction and coalescence, i.e.

\[ \dot{f} = \lambda(1-f)N_{kk} + \mathcal{A} \tilde{\varepsilon}^p - \frac{\mu^2}{\sigma_0} \nabla \cdot (\Delta \lambda \nabla f). \tag{35} \]

In order to specialize the variational statement shown in Eq. (34) to the case of the gradient-dependent Gurson model, we start by first integrating the evolution equations for \( \tilde{\varepsilon}^p \) and \( f \) over the time interval \([t_n, t_{n+1} = t_n + \Delta t]\) using a backward-Euler scheme to get

\[ (1 - f)\sigma_{ij} \Delta \tilde{\varepsilon}^p - \Delta \lambda \sigma : N, \tag{36} \]
\[ \Delta f = \Delta \lambda (1 - f)N_{kk} + \mathcal{A} \Delta \tilde{\varepsilon}^p - \frac{\mu^2}{\sigma_0} \nabla \cdot (\Delta \lambda \nabla f) \tag{37} \]

where all quantities are understood to be evaluated at time \( t_{n+1} \).

The variational statement is then written as:

Find \( u(x, t) \in H^1(V) \) satisfying \( u|_{S_p} = \bar{u} \), \( \lambda(x, t) \in H^1(V) \) satisfying \( \dot{\lambda} = 0 \) on \( S_p^0 \), \( \tilde{\varepsilon}^p(x, t) \in L^2(V) \) satisfying \( \tilde{\varepsilon}^p|_{S_p^0} = \bar{\varepsilon}^p \), \( f(x, t) \in H^1(V) \) satisfying \( f|_{S_p^0} = \bar{f} \) such that for all \( v^* \in L^2(V) \) satisfying \( v^*|_{S_p} = 0 \), for all \( \lambda^* \in L^2(V) \), for all \( q_1^*, q_2^* \in L^2(V) \), and for all \( a_\alpha^* \in L^2(V) \) satisfying \( a_\alpha^*|_{S_p} = 0 \),

\[ A(u, \lambda, \tilde{\varepsilon}^p, v^*, \lambda^*, q_1^*, q_2^*, a_\alpha^*) = \int_V (\sigma_{ij,j} + b_i) v_i^* \, dV - \int_S (\sigma_{ij} - \hat{\lambda}_{ij} v_i^* \, dS \]
\[ + \int_{\partial V} (q - \sigma_{ij} a_i) \lambda^* v_i^* \, dV + \int_{\partial V} [(1 - f)\sigma_{ij} \Delta \tilde{\varepsilon}^p - \Delta \lambda \sigma_{ij} N_{ij}] q_i^* \, dV \]
\[ + \int_{\partial V} \left[ \Delta f - \Delta \lambda (1 - f)N_{kk} - \mathcal{A} \Delta \tilde{\varepsilon}^p + \frac{\mu^2}{\sigma_0} \nabla \cdot (\Delta \lambda \nabla f) \right] q_2^* \, dV \]
\[ + \int_{S_p} f, \bar{p}_{ij} a_\alpha^* \, dS = 0, \tag{38} \]

where \( \sigma = \sigma(u, \lambda, \tilde{\varepsilon}^p, f) \), \( \alpha = \{1 + f^2 - 2\mu \cosh[3\mu / (2\sigma_m)]\}^{1/2} \),

\[ \sigma_{ij} N_{ij} = \frac{\partial \Phi}{\partial p} + q \frac{\partial \Phi}{\partial q} \quad \text{and} \quad N_{kk} = -\frac{\partial \Phi}{\partial p}. \tag{39} \]

Using Green's theorem and setting \( \alpha_\alpha^* = -(\tilde{\varepsilon}^2 / \sigma_0) \Delta \lambda q_\alpha^* \) results in the following variational statement:

Find \( u(x, t) \in H^1(V) \) satisfying \( u|_{S_p} = \bar{u} \), \( \lambda(x, t) \in H^1(V) \) satisfying \( \dot{\lambda} = 0 \) on \( S_p^0 \), \( \tilde{\varepsilon}^p(x, t) \in L^2(V) \) satisfying \( \tilde{\varepsilon}^p|_{S_p^0} = \bar{\varepsilon}^p \), \( f(x, t) \in H^1(V) \) satisfying \( f|_{S_p^0} = \bar{f} \) such that for all \( v^* \in H^1(V) \) satisfying \( v^*|_{S_p} = 0 \), for all \( \lambda^* \in L^2(V) \), for all \( q_1^* \in L^2(V) \), for all \( q_2^* \in H^1(V) \) satisfying \( q_2^*|_{S_p} = 0 \),
\[ A = \int_V (\sigma_j D_{ij}^* - b_j \nu_j^*) \, dV - \int_S \Gamma_v \nu_j^* \, dS \]
\[ - \int_V \left( q - \sigma_m \alpha \lambda^* \right) \, dV - \int_V \left[ (1 - f) \sigma_m \Delta \bar{\varepsilon}^p - \Delta \lambda \left( p \frac{\partial \Phi}{\partial p} + q \frac{\partial \Phi}{\partial q} \right) \right] q^*_i \, dV \]
\[ - \int_V \left[ \Delta f + \Delta \lambda(1 - f) \frac{\partial \Phi}{\partial p} - \alpha \Delta \bar{\varepsilon}^p \right] q^*_2 \, dV + \int_V \frac{\varepsilon^2}{\sigma_0} \Delta \lambda \nabla \cdot \nabla q^*_2 \, dV = 0, \quad (40) \]

where \( \sigma = \sigma(u, \lambda, \bar{\varepsilon}^p, f) \) and \( D_{ij}^* = (v_{i,j}^* + v_{j,i}^*)/2 \).

5.1. Finite element formulation

In the context of the formulation discussed above, the approximations used for the displacement \( u \), the plastic multiplier \( \lambda \), the equivalent plastic strain \( \bar{\varepsilon}^p \), and porosity \( f \) fields must be continuous, i.e. in \( C^0(V) \). The domain \( V \) is discretized, and standard element interpolations are introduced. The solution is obtained incrementally, and within each element we write at every instant

\[ \{u(x)\} = [N(x)]\{w_v\}, \quad \lambda = [h_1(x)]\{w_\lambda\}, \quad (41) \]
\[ \bar{\varepsilon}^p = [h_2(x)]\{w_{\varepsilon^p}\}, \quad f = [h_3(x)]\{w_f\}, \quad (42) \]

and

\[ \{v^*(x)\} = [N(x)]\{w^*_{v}\}, \quad \lambda^* = [h_1(x)]\{w^*_\lambda\}, \quad (43) \]
\[ q^*_1 = [h_2(x)]\{w^*_1\}, \quad q^*_2 = [h_3(x)]\{w^*_2\}, \quad (44) \]

where \([N(x)], [h_1(x)], [h_2(x)] \) and \([h_3(x)] \) are arrays containing standard element shape functions. \( \{w_v\} \) and \( \{w^*_{v}\} \) are the vectors of nodal quantities of the form

\[ \{w_v\} = [u_1, u_2, u_3, \lambda^*, \bar{\varepsilon}^p, f, \ldots, u_1^{\text{Nnode}}, u_2^{\text{Nnode}}, u_3^{\text{Nnode}}, \lambda^{\text{Nnode}}, \bar{\varepsilon}^{\text{Nnode}}, f^{\text{Nnode}}], \]
\[ \{w^*_{v}\} = [v^*_1, v^*_2, v^*_3, \lambda^*, q^*_1, q^*_2, \ldots, v_1^{\text{Nnode}}, v_2^{\text{Nnode}}, v_3^{\text{Nnode}}, \lambda^{\text{Nnode}}, q_1^{\text{Nnode}}, q_2^{\text{Nnode}}], \]

where \( \text{Nnode} \) is the number of nodes per element, and the notation \([a] = [a]^T\) is used. Using the above equations we can readily write

\[ \{D(x)\} = [B(x)]\{w_v\}, \quad \{D^*(x)\} = [B(x)]\{w^*_{v}\}. \quad (47) \]

and

\[ \{
abla \lambda(x)\} = [n_1(x)]\{w_\lambda\}, \quad \{
abla \lambda^*(x)\} = [n_1(x)]\{w^*_\lambda\}, \quad (48) \]
\[ \{
abla (x)\} = [n_3(x)]\{w_f\}, \quad \{
abla q^*_2(x)\} = [n_3(x)]\{w^*_2\}. \quad (49) \]

Let \( \{w\} \) and \( \{w^*\} \) be the corresponding global vectors of nodal unknowns. Substituting the above equations into the variational equation (40), and taking into account that the resulting equation must be satisfied for arbitrary \( \{w^*\} \), we arrive at a non-linear equation for \( \{w\} \) of the form

\[ \{G(\{w\})\} = \{0\}, \quad (50) \]

where
\[ \{G\} = \sum_{e=1}^{\text{NELEM}} \left( \int_{V_e} [B]^T \{\alpha\} \, dV - \int_{V_e} [N]^T \{\beta\} \, dV - \int_{S_e} [N]^T \{\tilde{\sigma}\} \, dS \right) \\
- \int_{V} \left\{ (q - \sigma_m) \{h_1\} \, dV - \int_{V} \left\{ (1 - f) \sigma_m \Delta \tilde{\epsilon}^p + \Delta \lambda \left( p \frac{\partial \phi}{\partial p} + q \frac{\partial \phi}{\partial q} \right) \right\} \{h_2\} \, dV \right. \\
- \left. \int_{V} \left[ \Delta f + \Delta \lambda (1 - f) \frac{\partial \phi}{\partial p} \right] \{h_3\} \, dV + \int_{V} \left[ \frac{\epsilon^p}{\sigma_0} \Delta \lambda \{n_1\}^T \{\lambda\} \, dV \right] \right) , \]

where the subscript \( e \) denotes the element-number, and \( \text{NELEM} \) is the total number of elements in the finite element mesh. The last equation is a nonlinear equation that must be solved for the nodal unknowns \( \{w\} \).

### 5.2. Numerical integration of the constitutive equations

In a finite element environment, the solution is constructed incrementally with the integration of the constitutive equations being performed at the elemental Gauss points. At a given material point, the solution \( (F_n, \sigma_n, \lambda_n, \tilde{\epsilon}_n, f_n) \) at time \( t_n \) as well as the deformation gradient \( F_{n+1} = \Delta F \cdot F_n \), and the increments \( \Delta \lambda, \Delta \tilde{\epsilon}^p \) and \( \Delta f \) corresponding to the time increment \( [t_n, t_{n+1} = t_n + \Delta t] \) are known, and the objective is to determine the stress \( \sigma_{n+1} \).

It is to be noted that, similar to the situation in Part I, the yield condition and evolution equations for the state variables are satisfied through variational statement and therefore do not figure explicitly along with the other rate equations to be handled at the material level. Likewise as in Part I, the values of \( \Delta \lambda \) and \( \Delta \tilde{\epsilon}^p \) at each node are required to be non-negative and the distinction between plastic loading \( (\Delta \lambda > 0) \) and elastic response \( (\Delta \lambda = 0) \) is made as described in Section 5.3 that follows.

More specifically, the constitutive equations to be integrated are as follows:

\[ D = D^e + D^p , \]
\[ \tilde{\sigma} = C^e : D^e , \]
\[ D^p = \Delta N , \quad N = \frac{\partial \Phi}{\partial \sigma} (\sigma, s_n) , \]

where

\[ C^e = \left( K - \frac{2}{3} G \right) \hat{I} + 2G \hat{g} \]

with \( K \) and \( G \) denoting the elastic bulk and shear moduli, respectively, and \( \hat{I} \) and \( \hat{g} \) representing the second- and fourth-order identity tensors, respectively. Likewise as in Part I, under the assumption that the Lagrangian triad associated with the incremental deformation gradient \( \Delta F(t) \) remains fixed in the time interval \( [t_n, t_{n+1}] \), the constitutive equations (52)-(54) can be written as

\[ E = E^e + E^p , \]
\[ \tilde{\sigma} = C^e : E^e , \]
\[ \tilde{E}^p = \Delta \hat{N} , \quad \hat{N} = \frac{\partial \Phi}{\partial \sigma} (\sigma, s_n) . \]

which are analogous to those of a ‘small-strain’ theory.

In the remainder of this section, we discuss the procedure to perform the numerical integration of the finite strain equations (56)-(58) along lines similar to ones outlined in [1]. Integration of (57) gives

\[ \tilde{\sigma}_{n+1} = \sigma_n + C^e : \Delta E = \sigma_n + C^e : (\Delta E - \Delta E^p) = \tilde{\sigma}^e - C^e : \Delta E^p , \]

where \( \tilde{\sigma}^e = \sigma_n + C^e : \Delta E \) is the (known) elastic predictor in co-rotational stress space. The backward Euler method is used to integrate the plastic flow rule (58):
\[
\Delta \mathbf{E}^p - \Delta \lambda \left( \frac{\partial \Phi}{\partial \mathbf{\sigma}} \right)_{n+1} - \Delta \lambda \left( -\frac{1}{3} \frac{\partial \Phi}{\partial \mathbf{p}} \mathbf{I} + \frac{\partial \Phi}{\partial \mathbf{q}} \mathbf{\dot{n}} \right)_{n+1},
\]

where

\[
\mathbf{n}_{n+1} = \frac{3}{2} \left( \frac{\dot{\mathbf{\sigma}}'}{\mathbf{q}} \right)_{n+1},
\]

\(\dot{\mathbf{\sigma}}'_{n+1}\) being the deviatoric part of \(\mathbf{\dot{\sigma}}_{n+1}\). In deriving Eq. (60) we took into account that the invariants \(p\) and \(q\) of \(\mathbf{\sigma}\) are the same as those of \(\dot{\mathbf{\sigma}}\).

Furthermore, introducing the notation

\[
\Delta \mathbf{E}_p = \Delta \lambda \left( \frac{\partial \Phi}{\partial \mathbf{p}} \right)_{n+1} \quad \text{and} \quad \Delta \mathbf{E}_q = \Delta \lambda \left( \frac{\partial \Phi}{\partial \mathbf{q}} \right)_{n+1},
\]

in (60), we have the expression for the incremental logarithmic plastic strain given by

\[
\Delta \mathbf{E}^p = \frac{1}{3} \Delta \mathbf{E}_p \mathbf{I} + \Delta \mathbf{E}_q \mathbf{n}_{n+1}.
\]

Substituting the expressions for \(\mathbf{C}^e\) and \(\Delta \mathbf{E}^p\) in (59), we find

\[
\mathbf{\dot{\sigma}}_{n+1} = \mathbf{\dot{\sigma}}^e - K \Delta \mathbf{E}_p \mathbf{I} - 2G \Delta \mathbf{E}_q \mathbf{n}_{n+1}.
\]

Considering the deviatoric part of the last equation, we can show readily that [1]

\[
\mathbf{n}_{n+1} = \frac{3 \mathbf{\dot{\sigma}}^e}{2q^e} = \text{known},
\]

where \(q^e = \sqrt{(3/2) \mathbf{\dot{\sigma}}^e : \mathbf{\dot{\sigma}}^e}\). and \(\mathbf{\dot{\sigma}}^e\) is the deviatoric part of \(\mathbf{\dot{\sigma}}\).

With \(\mathbf{n}_{n+1}\) known, Eq. (64) makes it clear that the determination of \(\mathbf{\dot{\sigma}}_{n+1}\) reduces to finding \(\Delta \mathbf{E}_p\) and \(\Delta \mathbf{E}_q\). Projecting Eq. (64) along \(\mathbf{I}\) and \(\mathbf{n}\), respectively, and taking into account that \(\mathbf{\dot{\sigma}}_{n+1} : \mathbf{I} = -3p_{n+1}\), \(\mathbf{\sigma}_{n+1} : \mathbf{n}_{n+1} = q_{n+1}\), and \(\mathbf{n}_{n+1} : \mathbf{n}_{n+1} = 3/2\) we find

\[
p = p^e + K \Delta \mathbf{E}_p \quad \text{and} \quad q = q^e - 3G \Delta \mathbf{E}_q,
\]

where \(p^e = -\mathbf{\dot{\sigma}}^e : \mathbf{I}/3\).

The unknowns \(\Delta \mathbf{E}_p\) and \(\Delta \mathbf{E}_q\) are determined as follows. Using the definitions (62), evaluating the derivatives \(\partial \Phi/\partial \mathbf{p}\) and \(\partial \Phi/\partial \mathbf{q}\), and using the expressions (66) for \(p\) and \(q\), we find

\[
\Delta \mathbf{E}_p = -\frac{3f \Delta \lambda}{\sigma_m} \sinh \left[ \frac{3(p^e + K \Delta \mathbf{E}_p)}{2\sigma_m} \right] \quad \text{and} \quad \Delta \mathbf{E}_q = \frac{2 \Delta \lambda}{\sigma_m^2} (q^e - 3G \Delta \mathbf{E}_q).
\]

Eq. (67a) is a nonlinear algebraic equation for \(\Delta \mathbf{E}_p\), which is determined by using Newton’s method, whereas (67b) can be solved for \(\Delta \mathbf{E}_q\) to give

\[
\Delta \mathbf{E}_q = \frac{2 \Delta \lambda q^e}{\sigma_m^2 + 6 \Delta \lambda G} = \text{known}.
\]

With \(\Delta \mathbf{E}_p\), \(\Delta \mathbf{E}_q\) and \(\mathbf{n}_{n+1}\) known, \(\mathbf{\dot{\sigma}}_{n+1}\) is determined from (64). Finally, the true stress at the end of the increment \(\mathbf{\sigma}_{n+1}\) is found from

\[
\mathbf{\sigma}_{n+1} = \mathbf{R}_{n+1} \cdot \mathbf{\dot{\sigma}}_{n+1} \cdot \mathbf{R}^T_{n+1},
\]

where \(\mathbf{R}_{n+1}\) is the rotation tensor associated with \(\Delta \mathbf{F}_{n+1}\).
5.3. A note on plastic loading/unloading

Part I discusses the procedure for developing and enforcing the discrete Kuhn–Tucker conditions which are appropriate for the present finite element formulation. For purposes of completeness, we will briefly outline the Kuhn–Tucker conditions in the context of the modified Gurson model.

In a continuum formulation, the Kuhn–Tucker conditions
\[ \Delta \lambda(x) \geq 0, \quad \Phi(x) \leq 0, \quad \Delta \lambda(x) \Phi(x) = 0, \]
must be satisfied at every point of the continuum.

In the present formulation, since the yield condition and evolution equations for the state variables are enforced globally, the loading/unloading conditions are also enforced in a ‘global’ sense.

We start with the finite element interpolation (41b) for \( \Delta \lambda \) and isolate the nodal degrees of freedom of the element that refer to \( \Delta \lambda \), denote the corresponding column-vector by \( \{ \Delta \lambda^N \} \), and rewrite (41b) within each element as
\[ \Delta \lambda(x) = \{ \tilde{h}_i(x) \} \{ \Delta \lambda^N \} \]
where \( \{ \tilde{h}_i(x) \} \) is the row-vector of shape functions that are used in the interpolation of \( \Delta \lambda(x) \). We also use the notation \( \{ \Delta \lambda^N \} \) to denote the corresponding global vector that contains all nodal degrees of freedom in the structure that refer to \( \Delta \lambda \).

Next, we define the global ‘nodal vector of the yield function’ \( \{ \overline{\Phi} \} \) so that
\[ \int_{\Omega} \Delta \lambda(x) \Phi(x) \, dV = [\Delta \lambda^N] \{ \overline{\Phi} \}, \]
which implies that
\[ \{ \overline{\Phi} \} = \sum_{e=1}^{\text{NELEM}} \{ \Phi_e^N \} \quad \text{where} \quad \{ \Phi_e \} = \int_{\Omega_e} \{ \tilde{h}_i(x) \} \Phi(x) \, dV, \]
and the notation of Part I is used.

The shape functions in (71) are chosen so that
\[ \{ \tilde{h}_i(x) \} \geq \{ 0 \}, \]
where the notation \( \{ a \} \geq \{ 0 \} \) means that all the components of \( \{ a \} \) are non-negative.

Using a procedure similar to that of Part I, we find that the discrete Kuhn–Tucker conditions can be written at every nodal point as follows:
\[ \Delta \lambda^N_\alpha \geq 0, \quad \overline{\Psi}_i \leq 0, \quad \Delta \lambda^N_i \overline{\Psi}_i = 0 \quad (no \ sum \ over \ i), \quad i = 1, \ldots, \text{NTOTAL}. \]

For the case of the gradient-dependent Gurson model discussed in Section 5, we have that
\[ \{ \overline{\Phi} \} = \int_{\Omega} \Delta \lambda(q - \sigma \alpha) \, dV \]
where \( \alpha = \{ 1 + \rho^2 - 2f \cosh[3p/(2\sigma \rho)] \}^{1/2} \). From the last equation \( \{ \overline{\Phi} \} \) can easily be concluded to be
\[ \{ \overline{\Phi} \} = \sum_{e=1}^{\text{NELEM}} \int_{\Omega_e} (q - \sigma \alpha) \{ \tilde{h}_i \} \, dV, \]
which is exactly the term in Eq. (51) that corresponds to the yield function.

In our finite element calculations, the discrete Kuhn–Tucker conditions are enforced as follows. The solution is determined incrementally, and, at every increment, each node is labeled as either ‘elastic’ or ‘plastic’. The condition \( \Delta \lambda = \Delta \varepsilon^p = \Delta f = 0 \) is enforced at all ‘elastic’ nodes, and the set (50) of nonlinear equations is solved for the nodal unknowns by using Newton’s method. Once a converged solution is obtained, that solution is accepted if
(1) the components of \( \{ \overline{\Phi} \} \) are such that \( \overline{\Psi}_i \leq 0 \) at all ‘elastic’ nodes, and
(2) the calculated nodal unknowns are such that \( \Delta \lambda \geq 0 \) and \( \Delta \varepsilon^p \geq 0 \) at all ‘plastic’ nodes.

If either of the above two conditions is violated at some nodes, then these nodes are relabeled and the solution
for the increment is repeated. This process is terminated, when an acceptable solution for the increment, i.e. one that satisfies both (1) and (2) above, is obtained.

5.4. Linearization of the finite element equations

The set of nonlinear equations (50) is solved for the vector of nodal unknowns \( \{w\} \), i.e. for the nodal values of \( u \), \( \lambda \), \( \varepsilon^p \) and \( f \), by using Newton’s method. In the following, we discuss briefly the determination of the corresponding Jacobian.

We start with the variational equation (40), i.e. \( A(u, \lambda, \varepsilon^p, f, v^*, \lambda^*, q_1^*, q_2^*) = 0 \), the linearization of which is expressed formally as

\[
A(u, \lambda, \varepsilon^p, f, v^*, \lambda^*, q_1^*, q_2^*) + DA(u, \lambda, \varepsilon^p, f, v^*, \lambda^*, q_1^*, q_2^*) \cdot (\Delta u, \Delta \lambda, \Delta \varepsilon^p, \Delta f) = 0,
\]

where

\[
DA(u, \lambda, \varepsilon^p, f, v^*, \lambda^*, q_1^*, q_2^*) = \frac{d}{d\varepsilon} A(u + \varepsilon, \lambda, \varepsilon^p + \varepsilon \Delta \varepsilon^p, f + \varepsilon \Delta f, v^*, \lambda^*, q_1^*, q_2^*) \bigg|_{\varepsilon = 0}.
\]

It can be shown easily that

\[
DA \cdot (du, d\lambda, d\varepsilon^p, df) = \int_V L^* : (d\sigma - \sigma \cdot d\mathbf{R} + d\mathbf{u} \otimes \mathbf{D}) \, dV - d\left( \int_{\partial V} (q - \alpha_\sigma \alpha) d\mathbf{\nu} \right)
\]

\[
- \int_V \left[ (1-f)\sigma_m \Lambda \varepsilon^p - \Lambda \left( \frac{\partial \Phi}{\partial \rho} + q \frac{\partial \Phi}{\partial q} \right) q_1^* \right] dV
\]

\[
- \int_V \left[ \Delta f (1-f) \frac{\partial \Phi}{\partial \rho} + \sigma_0 \Delta \varepsilon^p \right] q_2^* dV + \int_{\partial V} \frac{\sigma_0^2}{\sigma_0} \Delta \mathbf{\nu} \cdot \nabla q_2^* dV
\]

where

\[
L^* = \nabla v^* \quad \text{and} \quad d\mathbf{u} = \nabla (du).
\]

The quantity \( d\sigma \) plays a crucial role in the evaluation of the first term in the right-hand side of Eq. (80). A brief outline of the calculations is given in the following. Using the expression \( \sigma = R \cdot \mathbf{\dot{\sigma}} \cdot R^T \), we find

\[
d\sigma = (dR \cdot R^T) \cdot \sigma - \sigma \cdot (dR \cdot R^T) + R \cdot d\mathbf{\dot{\sigma}} \cdot R^T,
\]

where

\[
d\mathbf{\dot{\sigma}} = \frac{\partial \mathbf{\dot{\sigma}}}{\partial E} \frac{\partial E}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial u} + \frac{\partial \mathbf{\dot{\sigma}}}{\partial \mathbf{\lambda}} \frac{\partial \mathbf{\lambda}}{\partial \mathbf{u}} + \frac{\partial \mathbf{\dot{\sigma}}}{\partial \mathbf{\varepsilon}^p} \frac{\partial \mathbf{\varepsilon}^p}{\partial \mathbf{u}} + \frac{\partial \mathbf{\dot{\sigma}}}{\partial f} \frac{\partial \mathbf{\dot{\sigma}}}{\partial f} + \frac{\partial \mathbf{\dot{\sigma}}}{\partial \mathbf{\dot{\sigma}}}.
\]

The derivatives \( \partial \mathbf{\dot{\sigma}} / \partial E \), \( \partial \mathbf{\dot{\sigma}} / \partial \mathbf{\lambda} \), \( \partial \mathbf{\dot{\sigma}} / \partial \mathbf{\varepsilon}^p \), and \( \partial \mathbf{\dot{\sigma}} / \partial f \) are evaluated easily by using the expressions given in Section 5.2, where the algorithm for the integration of the elastoplastic constitutive equations is outlined, and are listed in Appendix A. The reader is referred to Part I for the derivation of the expressions for \( dR \cdot R^T \), \( \partial E / \partial \mathbf{u} \) and \( (\partial U_{\mathbf{u}} / \partial \mathbf{u}) \) du. The expression for \( d\sigma \) is finally given by substituting the expressions for \( dR \cdot R^T \) and \( d\mathbf{\dot{\sigma}} \) into (82) as follows:

\[
d\sigma = \Sigma : d\mathbf{u} + a \, d\mathbf{u} + b \, d\varepsilon^p + c \, df,
\]

where

\[
a_{ij} = R_{ik} \frac{\partial \mathbf{\dot{\sigma}}}{\partial \mathbf{\lambda}} R_{jl}, \quad b_{ij} = R_{ik} \frac{\partial \mathbf{\dot{\sigma}}}{\partial \varepsilon^p} R_{jl}, \quad c_{ij} = R_{ik} \frac{\partial \mathbf{\dot{\sigma}}}{\partial f} R_{jl},
\]

and the expression for \( \Sigma \) is also provided in Part I.
The second, third, fourth and fifth terms (differentials) in the right-hand side of Eq. (80) are evaluated in a way similar to that outlined in Appendix A of Part I; the details of these calculations can be found in [8] and will not be repeated here.

Substitution of these expressions and the expression for \( d\mathbf{r} \) into (80) with the subsequent introduction of the finite element interpolations results in

\[
DA(du, d\lambda, d\varepsilon^p, df) = [\mathbf{W^*}, [J][\mathbf{w}]],
\]

with \([J]\) being the desired Jacobian matrix.

6. Numerical examples

In this section, we present results from two example problems solved using the proposed scheme. The first example deals with the localization of plastic flow under plane strain tension, and the second is concerned with the problem of a mode-I plane strain crack.

The material used in our calculations is chosen to be elastic-plastic with the plastic behavior modeled using Gurson’s plasticity model. The yield stress of the matrix material \( \sigma_m(\varepsilon^p) \) is described by

\[
\sigma_m = \sigma_0 \left( 1 + \frac{\varepsilon^p}{\varepsilon_0} \right)^{1/N},
\]

where \( \sigma_0 \) is the initial tensile yield stress of the matrix material, \( \varepsilon_0 = \sigma_0/E \), and \( E \) is Young’s modulus.

The evolution equation of porosity is of the form

\[
\dot{f} = \dot{\lambda}(1-f)N_{kk} + \dot{\varepsilon}^p \varepsilon_0 - \frac{\varepsilon^p}{\varepsilon_0} (\dot{\lambda} \nabla^2 f + \nabla \dot{\lambda} \cdot \nabla f).
\]

The following values are used in the calculations: \( E = 300\sigma_0 \), \( N = 10 \), and \( \nu = 0.3 \), \( \nu \) being Poisson’s ratio. The plastic strain-controlled nucleation is described by the void volume fraction of void nucleating particles \( f_N = 0.1 \), the mean strain for nucleation \( \varepsilon_N = 0.3 \), and the standard deviation \( s_N = 0.1 \).

The results of our calculations are presented hereunder.

6.1. Localization under plane strain tension

The geometry used for the localization problem is the same as the one used in Part I. The aspect ratio of the specimen is chosen to be \( L/H = 2 \), where \( L \) and \( H \) are its original length and width, respectively. A schematic representation of one quarter of the specimen is shown in Fig. 2. Each material point in the specimen is identified by its position vector \( \mathbf{X} = (X_1, X_2) \) in the undeformed configuration. The specimen is subjected to plane strain tensile loading along the \( X_2 \) direction. The deformation is driven by the prescribed end displacement \( \mathbf{U} \), and the lateral surface on \( X_1 = H/2 \) is kept traction-free. As we are interested only in those solutions that are symmetric with respect to both the \( X_1 \) and \( X_2 \) coordinate directions, we analyze only one quadrant of the specimen as shown in Fig. 2. The applied boundary conditions are:

1. \( u_2 = 0, \sigma_{12} = 0 \) on \( X_2 = 0 \),
2. \( u_2 = 0, \sigma_{12} = 0 \) on \( X_1 = 0 \),
3. \( T_1 = T_3 = 0 \) on \( X_1 = H/2 \), and
4. \( u_2 = \dot{U} = \text{known}, \sigma_{12} = 0 \) on \( X_2 = L/2 \),

where \( T_1 \) and \( T_3 \) are the components of the nominal traction vector.

In order to trigger the initiation of non-homogeneous deformation in the specimen, small imperfections are introduced in the material properties. In particular, the initial porosity distribution \( f(t = 0) \) is assumed to vary in the specimen according to the expression

\[
f(t = 0) = f_0 \left[ 1 - 0.03 \left( \frac{X_2}{L/2} \right)^2 \right],
\]

where \( f_0 = 0.05 \). The above equation indicates that the initial porosity level is highest in the middle of the
Fig. 2. Schematic representation of one quarter of the specimen. A typical finite element mesh is also shown.

Fig. 3. Load-extension curves for the 'local' material (ε = 0).
Fig. 4. Contours of $f$ for the 'local' material ($\ell = 0$) at strain levels $\varepsilon^{ln} = 0.134, 0.159, 0.164$ and 0.168, calculated using the $24 \times 96$ mesh.
specimen \((X_2 = 0)\) where \(f(0) = f_0\), and lowest at the loading edge \((X_2 = L/2)\) where \(f(0) = 0.97f_0\). This makes the specimen initially ‘softer’ at its center \((X_2 = 0)\) and ‘harder’ at its loaded edge \((X_2 = L/2)\).

Four node isoparametric plane strain elements with \(2 \times 2\) integration points are used in the discretization. The initial (undeformed) finite element mesh is uniform in both directions. Four different meshes are used, namely \(8 \times 32, 16 \times 64, 24 \times 96\) and \(32 \times 128\) where the first and second numbers denote the number of elements along the \(X_1\) and \(X_2\) directions, respectively.

Two sets of calculations are carried out. The standard Gurson model \((\ell = 0)\) is used in the first set, whereas the second set is carried out using a gradient-dependent Gurson model with \(\ell = 0.2L\). For each of the two sets of calculations, solutions are obtained using all four different finite element meshes.

Figs. 3–6 show results corresponding to the ‘local’ Gurson model \((\ell = 0)\). Fig. 3 shows the ‘load-extension’ curves as calculated using the four different meshes. The normalized load \(F\) plotted in Fig. 3 is defined as \(F = F/(bH/2)\), where \(F\) is the calculated total axial force from the analysis of the one quarter of the specimen, and \(b\) its undeformed thickness. The solid curve in this figure corresponds to the homogeneous solution. It is found that the numerical solutions obtained using different mesh sizes agree with each other up to a macroscopic axial strain \(\varepsilon^m = 0.13\), where the ‘macroscopic axial strain’ is defined as \(\varepsilon^m = \ln[1 + U/(L/2)]\). At that strain level a shear band forms, and beyond that point the obtained numerical solutions exhibit a strong mesh-dependence, as is evident from the curves shown in Fig. 3. The contours shown in Fig. 4 demonstrate the evolution of \(f\) in the specimen up to the point of shear banding as calculated by using the \(24 \times 96\) mesh. It should be noted that, although only one quarter of the specimen is analyzed, the contour plots presented in the following are shown for the whole specimen. In the early stages, \(f\) is almost uniform with a very slight inhomogeneity due to the presence of initial imperfections. At the macroscopic strain level of \(\varepsilon^m = 0.13\), the inhomogeneity begins to grow into a shear band as shown in the figure. Fig. 5 shows the variation of \(f\) along \(X_1 = 0\) and \(X_1 = H/2\), at a macroscopic strain level of \(\varepsilon^m = 0.165\), as calculated using the four different meshes. The width of the shear band tends to zero as the mesh is refined, and the strong mesh dependence of the solution is clear. It should be noted, however, that this mesh dependence appears only beyond the critical strain \(\varepsilon^m = 0.13\). At smaller extension levels, the corresponding profiles of \(f\) are almost identical for all four different meshes. Fig. 6 shows contours of \(f\) at a macroscopic strain level \(\varepsilon^m = 0.165\), as calculated using the four different meshes. Once again, the strong mesh dependence of the solution is evident.

Figs. 7–9 show results for the case of a gradient-dependent Gurson material with \(\ell = 0.2L\). Fig. 7 shows ‘load-extension’ curves as calculated using the four different meshes. The solid line corresponds again to the homogeneous solution. The calculated load extension responses obtained using different mesh sizes are found to converge on a single curve as the mesh is refined. Fig. 8 shows the evolution profiles of \(f\) along \(X_1 = 0\) and

![Fig. 5. Variation of \(f\) for the ‘local’ material \((\ell = 0)\) along \(X_1 = 0\) and \(X_1 = H/2\) at a strain level \(\varepsilon^m = 0.165\), as calculated using the four different meshes.](image-url)
Fig. 6. Contours of $f$ for the 'local' material ($\ell = 0$) at a strain level $\varepsilon^{0} = 0.165$, as calculated using the four different meshes.
Fig. 7. Load-extension curves for the gradient-dependent Gurson material with $\ell = 0.2L$.

Fig. 8. Evolution of $f$ for the gradient-dependent Gurson material with $\ell = 0.2L$ along $X_1 = 0$ and $X_1 = H/2$ up to a strain level $\varepsilon^{ln} = 0.185$, as calculated using the four different meshes.
Fig. 9. Contours of $f$ for the gradient-dependent Gurson material with $\ell = 0.2L$ at a strain level $e^{\infty} = 0.185$, as calculated using the four different meshes.
Fig. 10. Finite element mesh used for the mode I crack under plane strain.

\[ X_i - \frac{H}{2}, \quad \text{up to a macroscopic strain level } \varepsilon^{\text{in}} = 0.185, \quad \text{as calculated using the four different meshes.} \]

The width of the shear band is now independent of the mesh size when the mesh is refined sufficiently. Fig. 9 shows contours of \( f \) at a macroscopic strain level \( \varepsilon^{\text{in}} = 0.185 \), as calculated using the four different meshes. Again, the contours shown in Fig. 9 make it clear that the solution is independent of the mesh size when the mesh is fine enough.

### 6.2. Mode I crack under plane strain

We consider the problem of ‘small scale yielding’ of a mode-I crack under plane strain conditions. Fig. 10 shows the finite element mesh used in the calculations, and Fig. 11 shows the details of the mesh in the region near the blunt crack tip. Because of symmetry, only the region above the crack plane is analyzed. A total of 1333 plane strain 4-node isoparametric elements with \( 2 \times 2 \) Gauss integration points are used in the calculations; the corresponding number of nodes is 1408.

The outermost radius of the mesh is \( R = 1500r_0 \), where \( r_0 \) is the initial radius of the circular notch. The symmetry conditions are enforced ahead of the crack plane, and displacements consistent with the leading term of the asymptotic expansion of the near-tip solution for a mode-I crack in an isotropic linear elastic material are applied on the outermost radius on the outer boundary of the mesh (e.g. see [2]). The applied displacement field is of the form

\[ u = \frac{K_I}{2G} \sqrt{\frac{R}{2\pi}} \tilde{u}(\nu, \theta), \]

where \( r \) and \( \theta \) are polar coordinates with the origin located at the center of curvature of the notch, \( K_I \) is the mode-I stress intensity factor, and \( \tilde{u} \) are dimensionless universal functions. The solution is developed incrementally as \( K_I \) increases from zero corresponding to mode-I loading.

The initial void volume fraction is taken to be uniform at 0.05. Two sets of calculations are carried out. One in which the ‘local’ form of the Gurson model is used (\( \ell = 0 \)), and another in which \( \ell = r_0 \).

Fig. 12 displays the variation of porosity ahead of the crack tip as the applied load increases from zero to \( K_I/(\sigma_0\sqrt{r_0}) = 25.51 \), for the case of the ‘local’ Gurson model (\( \ell = 0 \)). In that figure, \( x_i \) denotes the current position of a material point ahead of the crack tip, where the origin of the coordinate system is located at the

Fig. 11. Near-tip mesh used in the crack problem.
center of the undeformed notch. The corresponding evolution profiles for the gradient-dependent Gurson model with \( \ell = r_0 \) are shown in Fig. 13. It is noted that the gradient type Gurson model predicts a smaller concentration of porosity at the root of the notch and reduces the spatial gradient of \( f \) ahead of the crack. Detailed calculations of this problem, for different values of \( \ell \) are now underway and will be reported elsewhere.

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Appendix A. Expressions involved in \( \dot{\sigma} \)

The partial derivatives of \( \dot{\sigma} \) with respect to \( E, \lambda, \tilde{e}^p \), and \( f \) are evaluated by using Eq. (64) and are as follows:

\[
\frac{\partial \dot{\sigma}}{\partial E} = \mathcal{C}^{\epsilon} - K I \frac{\partial \Delta \tilde{e}_v}{\partial E} - 2G \left( \tilde{n} \frac{\partial \Delta \tilde{e}_v}{\partial E} + \Delta \tilde{e}_v \frac{\partial \tilde{n}}{\partial E} \right),
\]

(A.1)
\[ \frac{\partial \sigma}{\partial \lambda} = -K \frac{\partial \Delta E_p}{\partial \lambda} I - 2G \frac{\partial \Delta E_q}{\partial \lambda} \hat{n} , \]  
(A.2)

\[ \frac{\partial \sigma}{\partial \varepsilon^p} = -K \frac{\partial \Delta E_p}{\partial \varepsilon^p} I - 2G \frac{\partial \Delta E_q}{\partial \varepsilon^p} \hat{n} , \]  
(A.3)

\[ \frac{\partial \sigma}{\partial f} = -K \frac{\partial \Delta E_p}{\partial f} I - 2G \frac{\partial \Delta E_q}{\partial f} \hat{n} , \]  
(A.4)

where all quantities are understood to be evaluated at the end of the increment, i.e. at \( t = t_{n+1} \). The partial derivatives of \( \Delta E_p \) and \( \Delta E_q \) with respect to \( E, \lambda, \varepsilon^p \) and \( f \), are determined from Eqs. (67a) and (68):

\[ \frac{\partial \Delta E_p}{\partial E} = \frac{-9f}{2\sigma^2 + 9fK} \frac{\partial \rho^e}{\partial E} , \]  
(A.5)

\[ \frac{\partial \Delta E_q}{\partial E} = 2 \frac{\partial q^e}{\partial \varepsilon^p} \left( \frac{\sigma_m^2 + 6G \lambda}{\sigma_m^2 + 6G \Delta \lambda} \right) , \]  
(A.6)

\[ \frac{\partial \Delta E_p}{\partial \lambda} = -\frac{6f \sigma_m \sinh \omega}{2\sigma^2 + 9fK} \frac{\partial \rho^e}{\partial \lambda} , \]  
(A.7)

\[ \frac{\partial \Delta E_q}{\partial \lambda} = - \frac{2q^e}{\sigma_m^2 + 6G \Delta \lambda} \left( 1 - \frac{6G \lambda}{\sigma_m^2 + 6G \Delta \lambda} \right) , \]  
(A.8)

\[ \frac{\partial \Delta E_p}{\partial f} = \frac{f}{3\sigma_m^2 + 9fK} \frac{\partial \rho^e}{\partial f} , \]  
(A.9)

\[ \frac{\partial \Delta E_q}{\partial f} = - \frac{4 \Delta \lambda q^e \sigma_m}{(\sigma_m^2 + 6G \Delta \lambda)^2} \frac{\partial \rho^e}{\partial f} , \]  
(A.10)

\[ \frac{\partial \Delta E_p}{\partial f} = 6 \lambda \sigma_m \sinh \omega \]  
(A.11)

\[ \frac{\partial \Delta E_q}{\partial f} = \frac{2\sigma^2 + 9fK \lambda \cos \omega}{\sigma^2} \]  
(A.12)

with \( \omega = 3\rho/(2\sigma_m) \). The expressions for \( \partial \rho^e/\partial E, \partial q^e/\partial E \) and \( \partial \hat{n}/\partial E \) are given by

\[ \frac{\partial \rho^e}{\partial E} = -KI , \quad \frac{\partial q^e}{\partial E} = 2G \hat{n} \quad \text{and} \quad \frac{\partial \hat{n}}{\partial E} = \frac{3G}{q^e} \left( \hat{j} - \frac{2}{3} \hat{n} \hat{n} \right) . \]  
(A.13)

References


