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Analytical Lattice Boltzmann Solutions for Thermal Flow Problems

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ABSTRACT

Analytical solutions based on two 13-bit (hexagonal and square) and one 17-bit square lattice Boltzmann BGK models have been obtained for the Couette flow, with a temperature gradient at the boundaries. The analytical solutions for the unknown distributions functions are written as polynomials in powers of the space variable and the coefficients of the expansion are estimated in terms of characteristic flow quantities, the single relaxation time and the lattice spacing. The analytical solutions of the two 13-bit models contain some nonlinear deviations from the thermal hydrodynamic constraints and the analytical solutions, while the 17-bit square lattice model results into an exact representation of the nonisothermal Couette flow problem.

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I. INTRODUCTION

In the last few years the lattice Boltzmann method (LBM) has been developed into an effective computational scheme for a broad variety of thermo-fluid physical systems (Chen and Doolen, 1998). Since the appearance of the LBM and its predecessor, the lattice gas automata (LGA) some analytical solutions have been obtained for these methods for two and three dimensional flows (Cornubert et al., 1991; Henon, 1987; Luo, 1997; Luo et al., 1991). More recently, analytical solutions of the distribution functions for Poiseuille and Couette flows are found for the triangular and square LBM models (He et al., 1997; Zou et al., 1995). All these results allow one to calculate the viscosity from given collision rules, to improve the implemented boundary conditions and to justify the accuracy to expect from the method. Overall analytical LB approaches are enhancing our understanding of the method. Nevertheless no analytical solution has been previously reported for thermal flow problems. It is well known that one of the major shortcomings of the LBM is the lack of a satisfactory thermal model for heat transfer problems. One of the methodologies to develop thermal lattice Boltzmann models is the so-called "multi-speed approach" (Alexander et al., 1993; Chen et al., 1994). Although this approach has been shown to suffer from numerical instability (McNamara et al., 1995), some recent work has provided new alternatives and potential in this approach (Pavlo et al., 1998a; 1986b). Some of the velocity discretization models studied in previous work include 13-bit models for either the hexagonal (Alexander et al., 1993) or the square (Qian, 1993) grids, as well as typical 17-bit square lattices.

In the present work an investigation on the accuracy to expect from the aforementioned multi-speed models is attempted. The nonisothermal Couette flow is chosen as a typical thermal flow model problem and an analytical LBM formulation approach developed earlier (He et al., 1997; Zou et al., 1995) is extended to include heat transfer effects. Analytical expressions of the distribution functions are obtained and some guidance is given for thermal flow applications.

The well-known lattice Boltzmann evolution equation is given by

$$f_{\sigma,i}(\mathbf{x}+\mathbf{e}_{\sigma,i}\delta t, t+\delta t) - f_{\sigma,i}(\mathbf{x},t) = -\frac{1}{\tau}\left[f_{\sigma,i}(\mathbf{x},t) - f^{(0)}_{\sigma,i}(\mathbf{x},t)\right] \qquad (1)$$

where $f_{\sigma,i}(\mathbf{x},t)$ is the distribution function of the particle of type (σ, i) at position $\mathbf{x}$ and time t, $f^{(0)}_{\sigma,i}(\mathbf{x},t)$ is the corresponding equilibrium function of the particle, $\mathbf{e}_{\sigma,i}$ are the unit velocity vectors along the specified directions and τ is the single relaxation time, which controls the rate at which the system relaxes to the local equilibrium. All quantities in Eq. (1) are in nondimensional form (Sterling and Chen, 1996). The choice of $f^{(0)}_{\sigma,i}(\mathbf{x},t)$ is critical in thermal lattice Boltzmann models. To accurately simulate hydrodynamic phenomena Eulerian and Navier–Stokesian descriptions of real fluids must be fully recovered. In two dimensions this may be achieved by requiring that the moments of $f^{(0)}_{\sigma,i}(\mathbf{x},t)$ satisfy the relations (McNamara and Alder, 1993)

$$\sum_{\sigma,i} f^{(0)}_{\sigma,i} = n \tag{2a}$$

$$\sum_{\sigma,i} e_{\alpha,\sigma,i} f^{(0)}_{\sigma,i} = nu_\alpha \tag{2b}$$

$$\sum_{\sigma,i} e_{\alpha,\sigma,i} e_{\beta,\sigma,i} f^{(0)}_{\sigma,i} = nu_\alpha u_\beta + n\varepsilon\delta_{\alpha\beta} \tag{2c}$$

$$\sum_{\sigma,i} e_{\alpha,\sigma,i} e_{\beta,\sigma,i} e_{\gamma,\sigma,i} f^{(0)}_{\sigma,i} = nu_\alpha u_\beta u_\gamma + n\varepsilon(u_\alpha\delta_{\beta\gamma} + u_\beta\delta_{\alpha\gamma} + u_\gamma\delta_{\alpha\beta}) \tag{2d}$$

$$\sum_{\sigma,i} e^2_{\sigma,i} e_{\alpha,\sigma,i} e_{\beta,\sigma,i} f^{(0)}_{\sigma,i} = (u^2 + 6\varepsilon)nu_\alpha u_\beta + (u^2 + 4\varepsilon)n\varepsilon\delta_{\alpha\beta} \tag{2e}$$

where Greek subscripts indicate Cartesian components. For a thermal fluid taking into account the symmetry of the moments under exchange of any pairs of indices there are 13 such constrains in two dimensions (26 constraints in three dimensions). This suggests the need for at least 13 different particle velocities in order to guarantee the linear independence of the left hand side of Eq. (2) and the full recovery of the thermal Navier–Stokes equations up to the fourth order terms. Typical **AQ2** equilibrium function has the polynomial form (Pavlo et al., 1998b)

$$\begin{aligned} f^{(0)}_{\sigma,i} = n[&A_\sigma + B_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u}) + C_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u})^2 + D_\sigma u^2 + E_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u})^3 \\ &+ F_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u})u^2 + G_\sigma u^4 + H_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u})^2 u^2 + I_\sigma(\mathbf{e}_{\sigma,i}\cdot\mathbf{u})^4] \end{aligned} \tag{3}$$

where the coefficients are functions of the local density $n = \sum_{\sigma,i} f_{\sigma,i}$ and the internal energy $2n\varepsilon = \sum_{\sigma,i} f_{\sigma,i}(\mathbf{e}_{\sigma,i} - \mathbf{u})^2$, while the bulk velocity is defined by $n\mathbf{u} = \sum_{\sigma,i} f_{\sigma,i}\mathbf{e}_{\sigma,i}$. The form of expression (3) is based on a Taylor expansion of the Maxwellian equilibrium distribution in the local velocity $\mathbf{u}$ keeping terms up to the fourth power. The coefficients of the relaxation distribution (3) are obtained in such a manner to remove discrete lattice effects and consequently the resulting relaxation distribution is not the Maxwellian.

The Couette thermal-flow problem under investigation consists of a fluid contained between two plates, the upper one moving with velocity u_0

and the lower one is stationary, while a temperature difference exists between the boundaries. The x and y components of the velocity $u = (u_x, u_y)$ and normalized energy profiles must satisfy

$$u_x(y) = u_0 y \tag{4a}$$

$$u_y(y) = 0 \tag{4b}$$

and

$$\varepsilon(y) = \left[y + \frac{Br}{2} u_0^2 y(1-y)\right] \tag{4c}$$

respectively, where y is the normalized distance from the lower plate and the Brinkman number Br is the product of the Prandtl and Eckert numbers.

II. ANALYTICAL SOLUTIONS OF THE 13-BIT MODELS

First the 13-bit hexagonal lattice is considered. This model is consisting of one rest particle,

$$\mathbf{e}_{\sigma i} = 0, \tag{5a}$$

for $\sigma = 0$, and two nonzero speeds for which

$$\mathbf{e}_{\sigma i} = \sigma\left(\cos\frac{\pi(i-1)}{3},\ \sin\frac{\pi(i-1)}{3}\right), \tag{5b}$$

for $\sigma = 1, 2$ and $i = 1, 2, 3, 4, 5, 6$. Taking into account the constrains mentioned above one can easily solve for the unknown coefficients of the equillibrium distribution function. One possible solution is (Alexander et al., 1993):

$$\begin{aligned}
&A_0 = 1 - \frac{5}{2}\varepsilon + 2\varepsilon^2, \quad A_1 = \frac{4}{9}\varepsilon - \frac{4}{9}\varepsilon^2, \quad A_2 = -\frac{1}{36}\varepsilon + \frac{1}{9}\varepsilon^2,\\
&B_1 = \frac{4}{9} - \frac{4}{9}\varepsilon, \quad B_2 = -\frac{1}{36} + \frac{1}{9}\varepsilon,\\
&C_1 = \frac{8}{9} - \frac{4}{3}\varepsilon, \quad C_2 = -\frac{1}{72} + \frac{1}{12}\varepsilon,\\
&D_0 = -\frac{5}{4} + 2\varepsilon, \quad D_1 = -\frac{2}{9} + \frac{2}{9}\varepsilon, \quad D_2 = \frac{1}{72} - \frac{1}{18}\varepsilon,\\
&E_1 = -\frac{4}{27}, \quad E_2 = \frac{1}{108}, \quad F_1 = F_2 = 0.
\end{aligned} \tag{6}$$

At this point we note that using the above set of estimates for the coefficients only the zeroth, first, and second moments of the imposed constraints corresponding to Eqs. (2a), (2b), and (2c) respectively are satisfied, while for the third moments, given by Eq. (2d), cubic deviations are present. Actually none of the possible solutions satisfy exactly the required constraints.

Now suppose that there is a solution $f_{\sigma,i}(\mathbf{x},t)$ of Eq. (1) that exactly represents the Couette flow with a temperature gradient between the boundaries. The solution must contain the following properties:

$$f_{\sigma,i}(\mathbf{x},t) \text{ is time independent denoted by } f_{\sigma,i}(\mathbf{x}) \tag{7a}$$

$$f_{\sigma,i}(\mathbf{x}) \text{ is a function of one space variable denoted by } f_{\sigma,i}(y) \tag{7b}$$

$$\sum_{\sigma,i} f_{\sigma,i}(y) = n \tag{7c}$$

$$\sum_{\sigma,i} f_{\sigma,i}(y) e_{x,\sigma,i} = nu_x(y) = nu_0 y \tag{7d}$$

$$\sum_{\sigma,i} f_{\sigma,i}(y) e_{y,\sigma,i} = 0 \tag{7e}$$

$$\sum_{\sigma,i} f_{\sigma,i}(y) \frac{(\mathbf{e}_{\sigma,\mathbf{i}} - \mathbf{u}^2)}{2} = n\varepsilon(y) = n\left[y + \frac{Br}{2} u_0^2 y(1-y)\right] \tag{7f}$$

Equations (7a) and (7b) are due to the fact that the particular flow under investigation is steady and fully developed. Equations (7c–7f) are derived using the definitions of the local density, the x and y components of velocity and the internal energy respectively supplemented by the well-known analytical velocity and temperature profiles given in Eqs. (4).

Using properties (7a) and (7b) for $\sigma = 1, 2$ and $i = 0, 1, 4$, which correspond to the rest particle and the two particles with motion along the x-axis, Eq. (1) may be written as

$$f_{\sigma,i}(y) = f_{\sigma,i}(y) - \frac{1}{\tau}\left(f_{\sigma,i}(y) - f_{\sigma,i}^{(0)}(y)\right). \tag{8}$$

Hence for $\sigma = 1, 2$ and $i = 0, 1, 4$ we obtain $f_{\sigma,i}(y) = f_{\sigma,i}^{(0)}(y)$. To find the remaining distribution functions we note that the equilibrium distributions are functions of powers of y up to y^3 through linear dependence on the x-component of the velocity. Thus, the following form of the remaining unknown distribution functions is suggested:

$$f_{\sigma,i}(y) = n(a_{\sigma,i} + b_{\sigma,i}\, y + c_{\sigma,i}\, y^2 + d_{\sigma,i}\, y^3), \tag{9}$$

for $\sigma = 1, 2$ and $i = 2, 3, 5, 6$. The 32 unknown coefficients are estimated by implementing evolution Eq. (1) in all eight velocity directions accordingly. For example for $\sigma = 1$ and $i = 2$ we have

$$f_{1,2}(y+\delta) = f_{1,2}(y) - \frac{1}{\tau}(f_{1,2}(y) - f_{1,2}^{(0)}(y)), \tag{10}$$

where δ is the vertical spacing between the lattice rows. The expressions for $f_{1,2}(y)$ and $f_{1,2}^{(0)}(y)$ given by Eqs. (3) and (9) respectively, are substituted in Eq. (10) and then the left hand side of Eq. (8) is expanded using Taylor series. Equating terms of equal powers in y in the resulting equation leads to an algebraic system of linear equations to be solved for the unknown coefficients. Applying the same procedure to all directions for which the distribution function is unknown and solving the systems symbolically yields

$$\begin{aligned}
&a_{1,2} = a_{1,5} = A_1 - b_{1,2}\delta\tau - c_{1,2}\delta^2\tau - d_{1,2}\delta^3\tau,\\
&a_{1,3} = a_{1,6} = A_1 - b_{1,3}\delta\tau - c_{1,3}\delta^2\tau - d_{1,3}\delta^3\tau,\\
&a_{2,2} = a_{2,5} = A_2 - 2b_{2,2}\delta\tau - 4c_{2,2}\delta^2\tau - 8d_{2,2}\delta^3\tau,\\
&a_{2,3} = a_{2,6} = A_2 - 2b_{2,3}\delta\tau - 4c_{2,3}\delta^2\tau - 8d_{2,3}\delta^3\tau,\\
&b_{1,2} = -b_{1,5} = \frac{B_1}{2}u_0 - 2c_{1,2}\delta\tau - 3d_{1,2}\delta^2\tau,\\
&b_{1,3} = -b_{1,6} = -\frac{B_1}{2}u_0 - 2c_{1,3}\delta\tau - 3d_{1,3}\delta^2\tau,\\
&b_{2,2} = -b_{2,5} = B_2u_0 - 4c_{2,2}\delta\tau - 12d_{2,2}\delta^2\tau,\\
&b_{2,3} = -b_{2,6} = -B_2u_0 - 4c_{2,3}\delta\tau - 12d_{2,3}\delta^2\tau,\\
&c_{1,2} = c_{1,5} = \frac{C_1 + 4D_1}{4}u_0^2 - 3d_{1,2}\delta\tau,\\
&c_{1,3} = c_{1,6} = \frac{C_1 + 4D_1}{4}u_0^2 - 3d_{1,3}\delta\tau,\\
&c_{2,2} = c_{2,5} = (C_2 + D_2)u_0^2 - 6d_{2,2}\delta\tau,\\
&c_{2,3} = c_{2,6} = (C_2 + D_2)u_0^2 - 6d_{2,3}\delta\tau,\\
&d_{1,2} = -d_{1,5} = \frac{E_1}{8}u_0^3, \quad d_{1,3} = -d_{1,6} = -\frac{E_1}{8}u_0^3,\\
&d_{2,2} = -d_{2,5} = E_2u_0^3, \quad d_{2,3} = -d_{2,6} = -E_2u_0^3.
\end{aligned} \tag{11}$$

Putting these results into Eq. (9) we obtain analytical expressions for all 13 distribution functions. These analytical expressions are substituted finally into Eqs. (7c–7f) to find

$$\sum_{\sigma,i} f_{\sigma,i} = n \tag{12a}$$

$$\sum_{\sigma,i} e_{x,\sigma,i} f_{\sigma,i} = ny + \frac{1}{3} ny\tau(2\tau - 1)\delta^2 \tag{12b}$$

$$\sum_{\sigma,i} e_{y,\sigma,i} f_{\sigma,i} = 0 \tag{12c}$$

$$\sum_{\sigma,i} \frac{(\mathbf{e}_{\sigma,i} - \mathbf{u})^2}{2} f_{\sigma,i} = n\varepsilon - \left[\frac{1}{3} ny^2\tau(2\tau - 1) - \frac{2}{3} n\varepsilon\tau(2\tau - 1)\right]\delta^2 \tag{12d}$$

It is noticed that the conservation of mass and y-momentum equations are satisfied exactly, while there is a second order discrepancy in the x-momentum and energy equations. It is seen that the derived analytical solutions although represent an exact solution of the LB model, Eq. (1) do not represent an exact solution of the nonisothermal Couette flow problem. The analytical solution depends on τ and δ. It is seen that as δ goes to zero or τ approaches $1/2$ the exact solution is recovered.

Next the 13-bit square lattice is considered. This model has one rest particle

$$\mathbf{e}_{\sigma,i} = 0 \tag{13a}$$

for $\sigma = 0$ and three nonzero speeds for which

$$\mathbf{e}_{\sigma,i} = e_\sigma(\cos\beta_i, \sin\beta_i) \tag{13a}$$

for $\sigma = 1, 3$ and $i = 1, 2, 3, 4$ where $e_1 = 1$, $e_3 = 2$, $\beta_i = (i-1)\pi/2$ and

$$\mathbf{e}_{\sigma,i} = e_\sigma(\cos\beta_i, \sin\beta_i) \tag{13a}$$

for $\sigma = 2$ and $i = 1, 2, 3, 4$ where $e_2 = \sqrt{2}$, $\beta_i = (i - 1/2)\pi/2$. Using the constraints given by Eqs. (2) and the above set of discrete velocities we find the coefficients of the equilibrium function (3) to be

$$\begin{aligned}
&A_0 = \frac{1}{2}(2 - 5\varepsilon + 4\varepsilon^2), \quad A_1 = \frac{2}{3}(\varepsilon - \varepsilon^2), \quad A_3 = \frac{1}{24}(-\varepsilon + 4\varepsilon^2),\\
&B_1 = \frac{1}{3}(2 - 3\varepsilon), \quad B_2 = \frac{\varepsilon}{4}, \quad B_3 = \frac{1}{24}(-1 + 3\varepsilon),\\
&C_1 = \frac{1}{3}(2 - 3\varepsilon), \quad C_2 = \frac{1}{8}, \quad C_3 = \frac{1}{96}(-1 + 6\varepsilon),\\
&D_0 = -\frac{5}{4} + \varepsilon, \quad D_1 = \frac{1}{3}\varepsilon, \quad D_2 = \frac{1}{8}(-1 - 2\varepsilon), \quad D_3 = \frac{1}{24}\varepsilon,\\
&E_1 = \frac{1}{3}, \quad E_2 = \frac{1}{8}, \quad E_3 = \frac{1}{96}, \quad F_1 = -\frac{1}{2}, \quad F_2 = -\frac{1}{8},\\
&G_0 = \frac{1}{4}, \quad H_1 = -\frac{1}{6}, \quad H_3 = \frac{1}{96}.
\end{aligned} \tag{14}$$

The coefficients, which are not included in the above equations, are taken equal to zero. Higher order terms have been added to the expression of the equilibrium function and as a result in this case the first four moments of the equilibrium, given by Eqs. (7a–7d), are recovered. However, still it is not possible to find a solution to satisfy the fourth order constraints given by Eq. (2e).

Introducing Eqs. (7a) and (7b) we obtain $f_{\sigma,i}(y) = f^{(0)}_{\sigma,i}(y)$ for $\sigma = 1, 3$ and $i = 0, 1, 3$. In this case the remaining unknown distribution functions take the form

$$f_{\sigma,i}(y) = n(a_{\sigma,i} + b_{\sigma,i}\,y + c_{\sigma,i}\,y^2 + d_{\sigma,i}\,y^3 + e_{\sigma,i}\,y^4) \tag{15}$$

for $\sigma = 1, 2, 3$ and $i = 2, 3, 4, 6, 7, 8$. Following the same procedure as before we find

$$\begin{aligned}
a_{1,2} &= a_{1,4} = A_1 + c_{1,2}\delta^2\tau(-1+2\tau) + e_{1,2}\delta^4\tau(-1+8\tau-12\tau^2),\\
a_{2,1} &= a_{2,3} = A_2 - b_{2,1}\delta\tau - c_{2,1}\delta^2\tau - d_{2,1}\delta^3\tau - e_{2,1}\delta^4\tau,\\
a_{2,2} &= a_{2,4} = A_2 - b_{2,2}\delta\tau - c_{2,2}\delta^2\tau - d_{2,2}\delta^3\tau - e_{2,2}\delta^4\tau,\\
a_{3,2} &= a_{3,4} = A_3 + 4c_{3,2}\delta^2\tau(-1+2\tau) + 16e_{3,2}\delta^4\tau(-1+8\tau-12\tau^2),\\
b_{1,2} &= -b_{1,4} = -2c_{1,2}\delta\tau + 4e_{1,2}\delta^3\tau(-1+3\tau),\\
b_{2,2} &= -b_{2,4} = -B_2u_0 - 2c_{2,2}\delta\tau - 3d_{2,2}\delta^2\tau - 4e_{2,2}\delta^3\tau,\\
b_{2,1} &= -b_{2,3} = B_2u_0 - 2c_{2,1}\delta\tau - 3d_{2,1}\delta^2\tau - 4e_{2,1}\delta^3\tau,\\
b_{3,2} &= -b_{3,4} = -4c_{3,2}\delta\tau + 32e_{3,2}\delta^3\tau(-1+3\tau),\\
c_{1,2} &= c_{1,4} = D_1u_0^2 + 6e_{1,2}\delta^2\tau(-1+2\tau),\\
c_{2,1} &= c_{2,3} = (C_2+D_2)u_0^2 - 3d_{2,1}\delta\tau - 6e_{2,1}\delta^2\tau,\\
c_{2,2} &= c_{2,4} = (C_2+D_2)u_0^2 - 3d_{2,2}\delta\tau - 6e_{2,2}\delta^2\tau,\\
c_{3,2} &= c_{3,4} = D_3u_0^2 + 24e_{3,2}\delta^2\tau(-1+2\tau),\\
d_{2,1} &= -d_{2,3} = (E_2+F_2)u_0^3 - 4e_{2,1}\delta\tau,\\
d_{2,2} &= -d_{2,4} = -(E_2+F_2)u_0^3 - 4e_{2,2}\delta\tau,\\
d_{1,2} &= -d_{1,4} = -4e_{1,2}\delta\tau,\\
d_{3,2} &= -d_{3,4} = -8e_{3,2}\delta\tau,\\
e_{2,1} &= e_{2,3} = (G_2+H_2+I_2)u_0^4,\\
e_{3,2} &= e_{3,4} = G_3u_0^4,\\
e_{2,2} &= e_{2,4} = (G_2+H_2+I_2)u_0^4,\\
e_{1,2} &= e_{1,4} = G_1u_0^4.
\end{aligned} \tag{16}$$

Once the 13 distribution functions are estimated they are substituted back into Eqs. (7) to find that all properties are fully satisfied. As a result in the analytical solutions, based on the 13-bit square lattice model, all flow characteristics are recovered and the LB evolution equation is satisfied. Hence the solution is an exact representation of the thermal Couette flow problem and it is valid for any relaxation time τ and lattice spacing. The only pitfall is that the Navier–Stokes equations are not fully recovered since the fourth order constraints are not satisfied. This drawback is circumvented in the next session by proposing a 17 discrete velocity model.

The obtained results of the two 13-bit models are indicative for the accuracy to expect implementing the 13-bit hexagonal and square lattice. It is seen that the accuracy of the 13-bit square lattice is improved compared with the accuracy of the 13-bit hexagonal lattice. No remarks however, can be made regarding stability issues of the two models. Actually previous stability analysis performed on the two models (Pavlo et al., 1998b) has shown that the 13-bit hexagonal model is more stable than the 13-bit square model.

III. ANALYTICAL SOLUTIONS OF THE 17-BIT MODEL

The 17-bit model introduced here is a straightforward extension of the 9-bit model used in isothermal problems. It is consisting of one rest particle,

$$\mathbf{e}_{\sigma,i} = 0, \tag{17a}$$

for $\sigma = 0$, and four nonzero speeds for which

$$\mathbf{e}_{\sigma,i} = e_\sigma(\cos\beta_i, \sin\beta_i), \tag{17b}$$

for $\sigma = 1, 3$ and $i = 1, 2, 3, 4$ where $e_1 = 1$, $e_3 = 2$, $\beta_i = (i-1)\pi/2$ and

$$\mathbf{e}_{\sigma,i} = e_\sigma(\cos\beta_i, \sin\beta_i), \tag{17c}$$

for $\sigma = 2, 4$ and $i = 1, 2, 3, 4$ where $e_2 = \sqrt{2}$, $e_4 = 2\sqrt{2}$, $\beta_i = (i - 1/2)\pi$.

The analytical formulation has been described extensively in the previous section and so we will be brief here presenting only the new material. The coefficients of the 17-bit equilibrium distribution functions, expressed by Eq. (3) are

$$\begin{aligned}
&A_0 = \frac{1}{2}(2 - 5\varepsilon + 4\varepsilon^2), \quad A_1 = \frac{2}{3}(\varepsilon - \varepsilon^2), \quad A_3 = \frac{1}{24}(-\varepsilon + 4\varepsilon^2)\\
&B_1 = \frac{1}{3}(2 - 3\varepsilon), \quad B_2 = \frac{1}{4}\varepsilon, \quad B_3 = \frac{1}{24}(-1 + 3\varepsilon)\\
&C_1 = \frac{1}{3}(2 - 3\varepsilon), \quad C_2 = \frac{1}{12}(2 - 3\varepsilon),\\
&C_3 = \frac{1}{96}(-1 + 6\varepsilon), \quad C_4 = \frac{1}{384}(-1 + 6\varepsilon)\\
&D_0 = -\frac{1}{8}(5 + 7\varepsilon), \quad D_1 = \frac{1}{3}(-1 + 4\varepsilon),\\
&D_2 = -\frac{1}{2}\varepsilon, \quad D_3 = \frac{1}{48}(1 - \varepsilon), \quad D_4 = -\frac{1}{32}\varepsilon\\
&E_1 = -\frac{1}{3}, \quad E_2 = -\frac{1}{24}, \quad E_3 = \frac{1}{48}, \quad E_4 = \frac{1}{384},\\
&F_1 = \frac{1}{2}, \quad F_2 = -\frac{1}{8}, \quad F_3 = -\frac{1}{16},\\
&G_0 = \frac{1}{4}, \quad G_2 = \frac{1}{24}, \quad G_4 = -\frac{1}{96},\\
&H_1 = -\frac{1}{6}, \quad H_2 = -\frac{1}{24}, \quad H_3 = \frac{1}{96}, \quad H_4 = \frac{1}{384}
\end{aligned} \tag{18}$$

Again the coefficients, which are not included in the above expressions, are taken equal to zero. The set of equilibrium functions resulting from the above constants, unlike the ones obtained by the two 13-bit models, satisfy all 13 constraints given in Eq. (2).

The distribution functions for $\sigma = 1, 3$ and $i = 0, 1, 3$ are equal to the corresponding equilibrium distributions. The unknown distribution functions for the remaining directions have a polynomial form identical to Eq. (15). The unknown coefficients for $\sigma = 1, 3$ with $i = 2, 4$ and $\sigma = 2$ with $i = 1, 2, 3, 4$ are given by Eq. (16), while the coefficients for $\sigma = 4$ and $i = 1, 2, 3, 4$ are given by

$$\begin{aligned}
&a_{4,1} = a_{4,3} = A_4 - 2b_{4,1}\delta\tau - 4c_{4,1}\delta^2\tau - 8d_{4,1}\delta^3\tau - 16e_{4,1}\delta^4\tau,\\
&a_{4,2} = a_{4,4} = A_4 - 2b_{4,2}\delta\tau - 4c_{4,2}\delta^2\tau - 8d_{4,2}\delta^3\tau - 16e_{4,2}\delta^4\tau,\\
&b_{4,1} = -b_{4,3} = 2B_4u_0 - 4c_{4,1}\delta\tau - 12d_{4,1}\delta^2\tau - 32e_{4,1}\delta^3\tau,\\
&b_{4,2} = -b_{4,4} = 2B_4u_0 - 4c_{4,2}\delta\tau - 12d_{4,2}\delta^2\tau - 32e_{4,2}\delta^3\tau,
\end{aligned}$$

$$c_{4,1} = c_{4,3} = (4C_4 + D_4)u_0^2 - 6d_{4,1}\delta\tau - 24e_{4,1}\delta^2\tau,$$
$$c_{4,2} = c_{4,4} = (4C_4 + D_4)u_0^2 - 6d_{4,2}\delta\tau - 24e_{4,2}\delta^2\tau,$$
$$d_{4,1} = -d_{4,3} = 2(4E_4 + F_4)u_0^3 - 8e_{4,1}\delta\tau,$$
$$d_{4,2} = -d_{4,4} = -2(4E_4 + F_4)u_0^3 - 8e_{4,2}\delta\tau,$$
$$e_{4,1} = e_{4,3} = (G_4 + 4H_4 + 16I_4)u_0^4,$$
$$e_{4,2} = e_{4,4} = (G_4 + 4H_4 + 16I_4)u_0^4. \quad (19)$$

The resulting analytical distribution functions satisfy exactly all properties described by Eqs. (7) and the LB Eq. (1) and can be considered as an exact representation of the thermal Couette flow. The analytical solution is independent of τ and δ.

IV. CONCLUDING REMARKS

In conclusion we have developed analytical solutions of the thermal Couette flow with a temperature difference at the boundaries using three different discrete velocity models. The analytical solutions for the discretized distribution functions must satisfy all 13 thermal hydrodynamic constraints in order to ensure full recovery of the Navier–Stokes equations, the flow characteristics and the profiles of the macroscopic solutions. It is seen that the solutions based on a 17-bit square lattice fulfils all above requirements. The solution based on the 13-bit square lattice although represents exactly the thermal flow problem does not recover the fourth order constraint. Finally the solution based on the 13-bit hexagonal model posses a second order error in the thermal flow solutions and a third order deviation in the third order constraints.

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REFERENCES

Alexander, F. J., Chen, S., Sterling, J. D. (1993). Lattice Boltzmann thermohydrodynamics. *Phys. Rev. E* 47(4):R2249–R2252.

Chen, S., Doolen, G. D. (1998). Lattice Boltzmann method for fluid flows. *Annu. Rev. Fluid Mech.* 30:329–364.

Chen, Y., Ohachi, H., Akiyama, M. (1994). Thermal lattice Bhatnagar–Gross–Krook model without non-linear deviations in macrodynamic equations. *Phys. Rev. E* 50(3):2776–2783.

Cornubert, R., d'Humieres, D., Levermore, D. (1991). A Knudsen layer theory for lattice gases. *Physica D* 47(6):241–259.

He, X., Zou, Q., Luo, L. S., Dembo, M. (1997). Analytical solutions of simple flows and analysis of nonslip boundary conditions for the lattice Boltzmann BGK model. *J. Stat. Phys.* 87:115–136.

Henon, M. (1987). Viscosity of a lattice gas. *Complex Systems* 1:763–789.

Luo, L. S. (1997). Analytic solutions of linearized lattice Boltzmann equation. *J. Stat. Phys.* 88:913–926.

Luo, L. S., Chen, H., Chen, S., Doolen, G. D., Lee, Y. C. (1991). Generalized hydrodynamic transport in lattice-gas automata. *Phys. Rev. A* 43:7097–7100.

McNamara, G., Alder, B. (1993). Analysis of the lattice Boltzmann treatment of hydrodynamics. *Physica A* 194:218–228.

McNamara, G., Garcia, A. L., Alder, B. J. (1995). Stabilization of thermal lattice Boltzmann models. *J. Stat. Phys.* 81:395–408.

Pavlo, P., Vahala, G., Vahala, L. (1998a). Higher order isotropic velocity grids in lattice methods. *Phys. Rev. Lett.* 80(18):3960–3963.

Pavlo, P., Vahala, G., Vahala, L., Soe, M. (1998b). Linear stability analysis of thermo-lattice Boltzmann models. *J. Comput. Phys.* 139:79–91.

Qian, Y. H. (1993). Simulating thermohydrodynamics with lattice BGK models. *J. Sci. Comput.* 8(3):231–242.

Sterling, J., Chen, S. (1996). Stability analysis of lattice Boltzmann method. *J. Comput. Phys.* 123:196–206.

Zou, Q., Hou, S., Doolen, G. D. (1995). Analytical solutions of the lattice Boltzmann BGK model. *J. Stat. Phys.* 81:319–339.

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