ASYMPTOTIC ANALYSIS OF EXTRUSION OF POROUS METALS

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Abstract—Plane strain extrusion of fully dense and porous metals is analysed using asymptotic techniques. The extrusion die is assumed to taper gradually down the extrusion axis. The asymptotic expansions are based on a small parameter \( \epsilon \) which is defined as the ratio of the total reduction of the original cross-section to the length of the reduction region. Coulomb's law is used to model the frictional forces that develop along the metal–die interface and the coefficient of friction is assumed to be of order \( \epsilon \). Analytical solutions for the first two terms in the expansions are obtained. In the case of the fully dense metals, it is shown that the leading order \([O(1)]\) solution involves "slab flow." It is also shown that the next term in the expansion of the solution is \( O(\epsilon^2) \), and this provides a theoretical justification for the use of the so-called "slab methods" of analysis for dies of moderate slope. An asymptotic analysis of the extrusion of porous metals with dilute concentration of voids is also carried out. Gurson's plasticity model is used to describe the constitutive behavior of the material. The leading order solution is the same as that of the fully dense material and the effects of porosity enter as an \( O(\epsilon) \) correction. In order to verify the asymptotic solutions developed, detailed finite element calculations are carried out for both the fully dense and the porous material. The asymptotic solutions agree well with the results of the finite element calculations.

NOTATION

- \( \sigma_{ij}, \tilde{S}_{ij} \) Cauchy and deviatoric stress tensor
- \( \dot{\mathbf{v}} \) velocity vector
- \( D_{ij}^p \) plastic part of the deformation-rate tensor
- \( \lambda \) plastic multiplier
- \( L, \Delta h \) original length and reduction of the die
- \( h \) height of the die
- \( \sigma_o \) yield stress
- \( \epsilon \) ratio of the length to the reduction of the die
- \( \mu_c \) coefficient of friction at the die–metal interface
- \( f \) porosity
- \( \sigma_{m} \) microscopic equivalent stress in the matrix
- \( \tau_e \) microscopic equivalent plastic strain
- \( \dot{\varepsilon}_m \) macroscopic equivalent shear stress
- \( \dot{\mathbf{v}}_0 \) velocity of the incoming billet
- \( f_0 \) porosity of the entering billet
- \( \dot{\varepsilon}_0 \) Young's modulus
- \( \nu \) Poisson's ratio
- \( \dot{\varepsilon}_0 \) yield strain

1. INTRODUCTION

Extrusion is a process in which a billet of material is converted into a continuous product of uniform cross-section by forcing it through a suitably shaped die. The most commonly used methods of analysis of such forming processes are slip line solutions [1], upper bound techniques [2] and finite element methods [3–5].

The use of asymptotic methods has been suggested recently as an alternative technique for the solution of metal forming problems such as extrusion, drawing, and rolling. Sayir and Frommer [6] presented an application of the method of singular perturbations to problems of plane strain slip line fields in rigid perfectly plastic materials. Johnson [7] and Smet and Johnson [8] considered a rigid plastic material that obeys the von Mises yield criterion with associated flow rule and used asymptotic methods to analyse axisymmetric extrusion and plane strain rolling. Aravas and McMeeking [9] used a combination of
asymptotic and finite element techniques to analyse three-dimensional extrusion. Durban and Mear [10] using a “radial” velocity field presented an asymptotic solution for axisymmetric extrusion of porous metals.

An asymptotic analysis of plane strain extrusion of fully dense as well as porous metals is presented in this paper. Coulomb's law is used to model the frictional forces that develop along the metal–die interface and the effects of the coefficient of friction on the solution are examined in detail. The cross-section of the extrusion die is assumed to taper gradually down the extrusion axis. In practical situations the slope of the die is small when either the area reduction is small or the length of the die is large compared to the height of the original cross-section. Such smoothly tapering dies can be used for the fabrication of whisker reinforced metal–matrix composites, where the use of conventional conical dies leads to internal shearing and turbulence in the material thus causing fracture of the whiskers [11].

Our analysis covers both fully dense and porous metals. The latter could be a powder metallurgy product which is extruded in order to achieve a desired final shape and full density at the same time. Gurson’s [12] yield criterion with associated flow rule is used to describe the plastic behavior of the porous metal. Our asymptotic analysis is based on a small parameter \( \varepsilon \) defined as the ratio of the reduction to the length of the die. Using a regular perturbation method we obtain analytical expressions for the solution up to second order in \( \varepsilon \). It is shown that the leading order approximation involves “radial” flow, which provides a theoretical justification for the use of such velocity fields in the analysis of extrusion of gradually tapering dies. It is also found that the effects of elasticity enter the solution to order \( \varepsilon^2 \) or higher. Detailed finite element calculations are carried out both for a fully dense and a porous metal and comparisons with the predictions of the asymptotic solutions are made.

Standard notation is used throughout. Boldface symbols denote tensors, the order of which is indicated by the context, and the summation convention is used for repeated Latin indices.

2. FULLY DENSE METALS

2.1. Formulation of the problem

We consider plane strain extrusion of fully dense isotropic materials through a gradually tapering die. The shape of the die is given by the function \( \hat{y} = \hat{h}(\hat{x}) \), where \( \hat{x} \) is the direction of extrusion (Fig. 1). The dies taper at a maximum angle which is of order \( \varepsilon \) (\( \varepsilon \) being a small positive number), i.e.

\[
\frac{d\hat{h}}{d\hat{x}} = O(\varepsilon).
\]

![Fig. 1. Schematic representation of the reduction region of the die.](image)
The formulation of the problem is given in the following. The only non-zero stress and velocity components are $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \dot{v}_x$, and $\dot{v}_y$, where $z$ is the coordinate axis normal to the plane of deformation.

The equations of body-force-free quasi-static equilibrium are

$$\frac{\partial \sigma_{xx}}{\partial \dot{x}} + \frac{\partial \sigma_{xy}}{\partial \dot{y}} = 0,$$

and

$$\frac{\partial \sigma_{xy}}{\partial \dot{x}} + \frac{\partial \sigma_{yy}}{\partial \dot{y}} = 0.$$

The formulation presented in this section is for a rigid-perfectly-plastic material that yields according to the von Mises yield condition with associated flow rule. It is assumed that yielding occurs everywhere inside the reduction region of the die and the flow rule in that region is written as

$$\frac{\partial \dot{\sigma}_x}{\partial \dot{x}} = \dot{\lambda} \tilde{S}_{xx}, \quad \frac{\partial \dot{\sigma}_y}{\partial \dot{y}} = \dot{\lambda} \tilde{S}_{yy},$$

and

$$\frac{\partial \dot{\sigma}_x}{\partial \dot{y}} + \frac{\partial \dot{\sigma}_y}{\partial \dot{x}} = 2\dot{\lambda} \dot{\sigma}_{xy},$$

where $\dot{\lambda}$ is a non-negative plastic multiplier and $\tilde{S}_{ij}$ is the deviatoric part of the stress tensor. The condition of plane strain requires that $\tilde{S}_{zz} = 0$ and $\tilde{S}_{xx} = -\tilde{S}_{yy}$. Then the von Mises yield condition can be written as

$$\tilde{S}_{xx}^2 + \tilde{S}_{xy}^2 = \frac{\sigma_0^2}{3},$$

where $\sigma_0$ is the yield stress of the material.

A detailed discussion of the effects of elasticity on the solution is given in Appendix I where it is also shown that such effects enter the solution only to order $\epsilon^2$ or higher.

Coulomb's law is used to model the frictional effects along the metal–die interface. The boundary conditions of zero normal velocity and Coulomb's law of friction lead to the following expressions on $\gamma = \hat{h}(\dot{x})$;

$$\dot{\gamma} = \frac{d\hat{h}}{d\dot{x}} \dot{v}_x,$$

and

$$\dot{\sigma}_{xy} = \mu \left[ \dot{\sigma}_{yy} - 2 \frac{d\hat{h}}{d\dot{x}} \dot{\sigma}_{xy} + \left( \frac{d\hat{h}}{d\dot{x}} \right)^2 \dot{\sigma}_{xx} \right] + 2 \frac{d\hat{h}}{d\dot{x}} \tilde{S}_{xx} + \left( \frac{d\hat{h}}{d\dot{x}} \right)^2 \dot{\sigma}_{xy},$$

where $\mu$ is the coefficient of friction. The symmetry of the problem about the $x$ axis, implies that the shear stress and the transverse velocity vanish on $\gamma = 0$. A constant flux of material flowing through a given cross-section $\dot{x}$ leads to the following condition:

$$\int_0^{\hat{h}(\dot{x})} \dot{v}_x(\dot{x}, \gamma) d\gamma = \dot{h}_0 \dot{u}_0,$$

where $\dot{u}_0$ is the velocity of the entering material and $\dot{h}_0 = \dot{h}(0)$. The boundary condition that the net force at exit is zero requires that

$$\int_0^{\hat{h}_1} \dot{\sigma}_{xx}(\dot{L}, \gamma) d\gamma = 0,$$

where $\dot{h}_1 = \dot{h}(\dot{L})$, $\dot{L}$ being the length of the die (see Fig. 1).
Since the geometry of the die is varying gradually in the direction of extrusion, a simplification of the problem can be brought about by stretching one coordinate direction with respect to the other \[13\]. For slowly tapering dies \((\varepsilon = \text{small})\), \(y\) is normalized by the height reduction \(\Delta h\), and distance down the extrusion axis is made dimensionless by the length of the die \(L\), i.e.

\[
x = \frac{\xi}{L} \quad \text{and} \quad y = \frac{\eta}{\Delta h} = \frac{\eta}{\varepsilon L},
\]

where \(\varepsilon = \Delta h/L\). The following normalizations are also introduced:

\[
\begin{align*}
\sigma_{xx} &= \frac{\sigma_{xx}}{\sigma_0}, & \sigma_{yy} &= \frac{\sigma_{yy}}{\sigma_0}, & \sigma_{xy} &= \frac{\sigma_{xy}}{\varepsilon \sigma_0}, \\
v_x &= \frac{v_x}{u_0}, & v_y &= \frac{v_y}{\varepsilon u_0}, & \lambda &= \frac{\Delta h}{u_0}.
\end{align*}
\] (14)

It should be noted that \(\sigma_{xy}\) and \(v_y\) are normalized by \(\sigma_0\) and \(u_0\), respectively. It can be readily shown, however, that normalization by \(\sigma_0\) and \(u_0\) would lead to zero leading-order \([i.e. O(1)]\) \(\sigma_{xx}\) and \(v_y\), thus justifying the normalization used in Eqns (13) and (14).

The coefficient of friction is assumed to be \(O(\varepsilon)\) and we define

\[
\mu = \frac{\mu}{\varepsilon},
\]

where \(\mu\) is now \(O(1)\).

The governing differential equations and boundary conditions are now written in terms of the non-dimensional variables. The equilibrium equations are:

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0, \quad (16) \\
\varepsilon^2 \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0. \quad (17)
\end{align*}
\]

The flow rule and yield condition become

\[
\begin{align*}
\frac{\partial v_x}{\partial x} &= \lambda S_{xx}, \quad (18) \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0, \quad (19) \\
\frac{\partial v_x}{\partial y} + \varepsilon^2 \frac{\partial v_y}{\partial x} &= 2\varepsilon^2 \lambda \sigma_{xy}, \quad (20) \\
3(S_{xx}^2 + \varepsilon^2 \sigma_{xy}^2) &= 1, \quad (21)
\end{align*}
\]

where Eqns (4) and (5) have been combined to obtain Eqn (19).

The boundary conditions on the normalized boundary \(y = h(x)\) can be written as

\[
\begin{align*}
v_y &= h v_x, \quad (22) \\
\sigma_{xy} &= \mu \sigma_y + 2h'S_{xx} + \mu \varepsilon^2 h'(h' \sigma_{xx} - 2 \sigma_{xy}) + \varepsilon^2 h^2 \sigma_{xy}, \quad (23)
\end{align*}
\]

where \(\cdot\) denotes derivative with respect to \(x\). The boundary conditions (10) and (11) become

\[
\begin{align*}
\int_0^{h(x)} v_x(x, y) \, dy &= h_0, \quad (24) \\
\int_0^{h_1} \sigma_{xx}(1, y) \, dy &= 0, \quad (25)
\end{align*}
\]

where \(h_1 = h(1)\) \((x = 1\) at the exit in normalized coordinates).
Equations (16)-(25) show that the solution of the normalized problem is of the form

\( \mathbf{v} = \mathbf{v}(x, y, \mu, \varepsilon^2), \) 
\( \mathbf{\sigma} = \mathbf{\sigma}(x, y, \mu, \varepsilon^2), \)

and

\( \lambda = \lambda(x, y, \mu, \varepsilon^2). \)

2.2. Perturbation expansion

We seek a perturbation expansion in \( \varepsilon^2 \) for the solution to the problem, such that

\( \mathbf{v} = \mathbf{v}^{(0)} + \varepsilon^2 \mathbf{v}^{(2)} + O(\varepsilon^4), \)
\( \mathbf{\sigma} = \mathbf{\sigma}^{(0)} + \varepsilon^2 \mathbf{\sigma}^{(2)} + O(\varepsilon^4), \)
\( \lambda = \lambda^{(0)} + \varepsilon^2 \lambda^{(2)} + O(\varepsilon^4). \)

Substituting the above expansions into the non-dimensionalized problem and collecting terms having like powers of \( \varepsilon \) we obtain the following hierarchy of problems.

The leading order problem is

\( \frac{\partial \sigma^{(0)}_{xx}}{\partial x} + \frac{\partial \sigma^{(0)}_{yy}}{\partial y} = 0, \)
\( \frac{\partial \sigma^{(0)}_{xy}}{\partial y} = 0, \)
\( \frac{\partial \sigma^{(0)}_{x}}{\partial x} = \lambda^{(0)} S^{(0)}_{xx}, \)
\( \frac{\partial \sigma^{(0)}_{x}}{\partial y} = 0, \)
\( \frac{\partial \sigma^{(0)}_{y}}{\partial x} = 0, \)
\( 3S^{(0)}_{xx} = 1. \)

The boundary conditions on the die–metal interface \( y = h(x) \) become to leading order

\( v^{(0)}_y = h' v^{(0)}_x, \)
\( \sigma^{(0)}_{xy} = \mu \sigma^{(0)}_{yy} + 2h S^{(0)}_{xx}. \)

Finally, the conditions of mass conservation and zero net force at the exit are written to leading order as

\( \int_0^{h(x)} v^{(0)}_x(x, y) \, dy = h_0, \)
\( \int_0^{h} \sigma^{(0)}_{xx}(1, y) \, dy = 0. \)

At \( O(\varepsilon^2) \) the problem is given by

\( \frac{\partial \sigma^{(2)}_{xx}}{\partial x} + \frac{\partial \sigma^{(2)}_{yy}}{\partial y} = 0, \)
\( \frac{\partial \sigma^{(2)}_{xy}}{\partial y} = 0, \)
\( \frac{\partial \sigma^{(2)}_{x}}{\partial x} = \lambda^{(0)} S^{(2)}_{xx} + \lambda^{(2)} S^{(0)}_{xx}, \)
The boundary conditions to this order are as follows:

\[
\begin{aligned}
&v^{(2)}_y = h' v^{(2)}_x, \\
&\sigma^{(2)}_{xy} = \mu(\sigma^{(2)}_{yy} - 2h' \sigma^{(0)}_{xy} + h'^2 \sigma^{(0)}_{xx}) + 2h' S^{(2)}_{xx} + h'^2 \sigma^{(0)}_{xy},
\end{aligned}
\tag{48}
\]

and

\[
\begin{aligned}
&\int_0^{h(x)} v^{(2)}_x(x, y) \, dy = 0, \\
&\int_0^{h_1} \sigma^{(2)}_{xx}(1, y) \, dy = 0.
\end{aligned}
\tag{50}
\]

### 2.3. Solution to the problem

We begin solving the leading order problem by determining the velocity field. From Eqns (36) and (40) the longitudinal velocity can be determined as

\[
v^{(0)}_x(x) = h_0 \frac{h(x)}{h(x)}.
\tag{51}
\]

The above equation shows that the leading order solution involves "slab flow" as assumed by von Kármán [14] in the analysis of plane strain rolling. The leading order incompressibility condition (35) gives the transverse velocity as

\[
v^{(0)}_y(x, y) = h_0 \frac{h'(x)}{h^2(x)} y = v^{(0)}_x(x) \frac{h'(x)}{h(x)} y.
\tag{52}
\]

For the case of a straight die, Eqns (51) and (52) show that the leading order velocity field is "radial" [15] and provide a theoretical justification for the use of such velocity fields in the analysis of extrusion through dies with moderate slopes. It should also be noted that the flow of the material inside the die (be it straight or smoothly curved) is, to leading order, along the streamlines \( y = c h(x) \), where \( c \) varies from 0 on the x-axis to 1 on the die-metal interface.

The leading order yield condition (37) requires that

\[
S^{(0)}_{xx} = - S^{(0)}_{yy} = \frac{1}{\sqrt{3}},
\tag{53}
\]

and the flow rule (34) implies that

\[
\dot{\alpha}^{(0)}(x) = - \sqrt{3} h_0 \frac{h'(x)}{h^2(x)}.
\tag{54}
\]

The second of the equilibrium equations (33) shows that \( \sigma_{yy} \) is, to leading order, independent of \( y \), i.e. \( \sigma^{(0)}_{yy} = \sigma^{(0)}_{yy}(x) \). Also, using the relationship \( \sigma^{(0)}_{xx} - \sigma^{(0)}_{yy} = S^{(0)}_{xx} - S^{(0)}_{yy} \) we find that \( \sigma^{(0)}_{xx}(x) = \sigma^{(0)}_{yy}(x) + 2/\sqrt{3} \), and the first equilibrium equation (32) becomes

\[
\sigma^{(0)}_{yy}(x, y) = - \sigma^{(0)}_{yy}(x) y.
\tag{55}
\]

Using the friction boundary condition (39) on the die–metal interface we get the following differential equation for \( \sigma^{(0)}_{yy}(x) \):

\[
\sigma^{(0)}_{yy}(x) + \mu \frac{h(x)}{h(x)} \sigma^{(0)}_{yy}(x) = - \frac{2}{\sqrt{3}} \frac{h'(x)}{h(x)}.
\tag{56}
\]
The above equation can be integrated using the exit boundary condition (41), to give the following expression for the stress $\sigma_{yy}^{(0)}$ and, subsequently, the leading order shear stress:
\[
\sigma_{yy}^{(0)}(x) = \frac{2}{\sqrt{3}} \int_x^1 h(\xi) \exp \left( -\mu \int_x^1 \frac{d\eta}{h(\eta)} d\xi - 1 \right) \exp \left( \mu \int_x^1 \frac{d\eta}{h(\eta)} \right),
\]
\[
\sigma_{yy}^{(0)}(x, y) = \left[ \mu \sigma_{yy}^{(0)}(x) + \frac{2}{\sqrt{3}} h'(x) \right] \frac{y}{h(x)},
\]
which completes the solution of the leading order problem.

It is interesting to note that the normal stress components are, to leading order, uniform over any transverse section as assumed by Sachs [16] in his approximate analysis of sheet drawing. The shear stress $\hat{\sigma}_{xy}$, which is an order of magnitude smaller than the normal stress components, varies, to leading order, linearly on each cross-section from zero on $y = 0$ to a maximum value on the metal–die interface. It is also found that the presence of friction does not affect, to leading order, the velocity field or the deviatoric stress components, but it does affect the hydrostatic stress in an exponential way.

The solution to the second order problem is obtained in a similar way, and the results are summarized in the following. The velocity field is given as:
\[
v_1^{(2)}(x, y) = \frac{1}{2} F(x) \left[ y^2 - \frac{1}{2} h^2(x) \right],
\]
\[
v_2^{(2)}(x, y) = \frac{1}{2} \left[ F'(x) h^2(x) + 2 F(x) h'(x) h(x) - F'(x) y^2 \right] y,
\]
The stress field is found to be:
\[
\sigma_{xx}^{(2)}(x, y) = \frac{1}{2} \sigma_{yy}^{(0)}(x) - \frac{1}{2} \sigma_{yy}^{(0)}(1) y^2 - \sigma^{(0)}(x) y + g(x),
\]
\[
\sigma_{xy}^{(2)}(x, y) = \frac{1}{2} \sigma_{yy}^{(0)}(x) y^2 + g(x),
\]
\[
\sigma_{yy}^{(2)}(x, y) = \left[ 2 \sigma_{yy}^{(0)}(x) \sigma_{yy}^{(0)}(x) - \frac{1}{6} \sigma_{yy}^{(0)}(x) \right] y^3 - g'(x) y,
\]
where
\[
g(x) = \left[ c - \int_x^1 A(\xi) \exp \left( -\mu \int_x^1 \frac{d\eta}{h(\eta)} d\xi \right) \exp \left( \mu \int_x^1 \frac{d\eta}{h(\eta)} \right) \right],
\]
\[
g'(x) = A(x) - \frac{g(x)}{h(x)},
\]
\[
c = \left[ \frac{1}{\sqrt{3}} (\sigma_{yy}^{(0)}(1))^2 - \frac{1}{6} \sigma_{yy}^{(0)}(1) \right] h_1^2,
\]
and
\[
A(x) = \left[ \frac{2}{\sqrt{3}} h(x) \sigma_{yy}^{(0)}(x) + \sqrt{3} h'(x) \sigma_{yy}^{(0)}(x) \right] h(x) \sigma_{yy}^{(0)}(x) - \frac{1}{6} h^2(x) \sigma_{yy}^{(0)}(x)
\]
\[- \frac{1}{2} \mu h(x) \sigma_{yy}^{(0)}(x) + \frac{1}{2 \mu} h'(x) \sigma_{yy}^{(0)}(x) - \frac{1}{2 \mu} h'(x) \sigma_{yy}^{(0)}(x) \left[ \sigma_{yy}^{(0)}(x) + \frac{2}{\sqrt{3}} \right].
\]

Figures 2 and 3 show the variation of the leading order stress and velocity components in the reduction region for the case of a straight die, i.e. $h(x) = h_0 - \Delta h \xi / L$, with reduction ratio $r = \Delta h / h_0 = 0.25$ for different values of the coefficient of friction $\mu$. The second order solution for the same die geometry is shown in Figs 4–6. Figure 4 shows the longitudinal variation of the second order stresses at the die–metal interface $y = h(x)$; the transverse variation of these stresses at $\xi = L/2$ is shown in Fig. 5. Both longitudinal and transverse variations of the second order velocities are shown in Fig. 6.
It should be noted that for $\mu = 0$ and $\epsilon > 0.2$ the contribution of the second order terms in the stress expansion is more than 10%. Similarly, for $\mu = 0.1$ and $\epsilon > 0.07$ the second terms contribute more than 10%, which suggests that, in most practical cases, the $O(\epsilon^2)$ terms can have a substantial contribution to the solution and cannot be ignored.

It should also be noted that Johnson [7], has carried out an asymptotic analysis for the axisymmetric extrusion of a power-law hardening material, where he used a different expansion parameter $\beta$. In his analysis, which is valid for conical dies (analogous to straight dies in our case), Johnson showed that the corrections to the leading order problem are of order $\beta = O(\mu \epsilon)$. If $\mu$ is $O(\epsilon)$, then higher order effects enter at $\beta = O(\epsilon^2)$ in his analysis also.

The results shown in Figs 2–6 illustrate the effects of $\mu$ on the various stress and velocity components. It appears that the dependence of the second order stress components on $\mu$ is
Asymptotic analysis of extrusion of porous metals

1.50
-0.50
1.25
0.50
0.75
1.00
0.000 0.250 0.500 0.750 1.000

(a)

Fig. 3. Longitudinal variation of the leading order velocity components.

stronger than that of the leading order solution. The effect of friction is more pronounced in Fig. 6(c), where it is seen that as \( \mu \) increases \( v_x^{(2)} \) becomes negative on \( y = h(x) \) and positive along the axis of extrusion, thus inducing large shear deformation gradients on the die-metal interface.

Summarizing, we mention that the solution to the problem is of the form

\[
\sigma_{xx}(x, y) = \sigma_{00} \left[ \sigma_{xx}^{(0)}(x) + \varepsilon^2 \sigma_{xx}^{(2)}(x, y) + O(\varepsilon^4) \right], \quad (69)
\]

\[
\sigma_{yy}(x, y) = \sigma_{00} \left[ \sigma_{yy}^{(0)}(x) + \varepsilon^2 \sigma_{yy}^{(2)}(x, y) + O(\varepsilon^4) \right], \quad (70)
\]

\[
\sigma_{xy}(x, y) = \sigma_{00} \left[ - \varepsilon \sigma_{xy}^{(0)}(x) y + \varepsilon^3 \sigma_{xy}^{(2)}(x, y) + O(\varepsilon^5) \right], \quad (71)
\]

and

\[
\vartheta_x = \vartheta_0 \left[ h_0 \frac{h(x)}{h(x)} + \varepsilon^2 \vartheta_x^{(2)}(x, y) + O(\varepsilon^4) \right], \quad (72)
\]

\[
\vartheta_y = \vartheta_0 \left[ \varepsilon h_0 \frac{h'(x)}{h(x)} y + \varepsilon^3 \vartheta_y^{(2)}(x, y) + O(\varepsilon^5) \right]. \quad (73)
\]

3. POROUS METALS

In this section, we consider plane strain extrusion of a porous metal through a gradually tapering die. Gurson’s [12] pressure dependent plasticity model is used to describe the constitutive behaviour of the porous medium. A brief description of Gurson’s model is given in Section 3.1; the formulation and the solution to the problem are discussed in Sections 3.2–3.4.

3.1. Gurson’s plasticity model

Here we briefly describe Gurson’s plasticity model for a solid containing a dilute concentration of voids. Based on a rigid-plastic upper bound solution for spherically symmetric deformations of a single spherical void, Gurson [12] proposed yield condition of the following form

\[
\Phi = \xi^2 + \frac{\tilde{\sigma}^2}{3} \left[ 2f \cosh\left( \frac{3}{2} \frac{\beta}{\tilde{\sigma}_m} \right) - (1 + f^2) \right] = 0, \quad (74)
\]

where \( \xi = \sqrt{\frac{1}{2} S_{ij} S_{ij} } \), \( \beta = \frac{1}{2} \tilde{\sigma}_{kk} \) is the hydrostatic stress, \( \tilde{\sigma}_m \) is the microscopic equivalent stress in the matrix material and \( f \) is the void volume fraction. For a perfectly-plastic matrix material, \( \tilde{\sigma}_m = \tilde{\sigma}_0 = \text{constant} \).
For a fully dense medium \( \tilde{f} = 0 \), the above yield condition reduces to that of von Mises and is independent of pressure. The aforementioned yield condition is also used as a plastic potential and the plastic part of the deformation-rate tensor is written as

\[
\tilde{\Phi}_{ij} = \lambda \frac{\partial \Phi}{\partial \tau_{ij}}.
\]  

(75)

In Gurson's model the microscopic equivalent plastic strain \( \tilde{\varepsilon}^p \) in the matrix material is assumed to vary according to the equivalent plastic work expression

\[
(1 - \tilde{f})\tilde{\varepsilon}^m_{ij} = \tilde{\varepsilon}^p_{ij}.
\]

(76)
where the superposed dot indicates the material time derivative. The rate of change of porosity is related to the deformation rate as

\[ \dot{\mu} = (1 - \dot{f}) D_{kk} \cdot \]  

(77)

3.2. Formulation of the extrusion problem

The formulation of the problem for the porous metal is very similar to that outlined in Section 2.1 for the fully dense material. The matrix material is assumed to be rigid–perfectly-plastic, i.e. $\gamma_m = \dot{\gamma}_0$, and the initial porosity is $f_0$. We assume that the porosity in the metal is
\( O(\varepsilon) \) and define

\[ f = \frac{f}{\varepsilon} \quad \text{and} \quad f_0 = \frac{f_0}{\varepsilon}, \]

where \( f \) and \( f_0 \) are now \( O(1) \). Using the dimensionless quantities defined in Eqns (12)–(15), we can write the normalized problem as follows.

The equilibrium equations become

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0, \]

\[ \varepsilon^2 \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0. \]

The condition of plane strain requires that

\[ S_{xx} + S_{yy} = -S_{zz} = \frac{1}{2} \varepsilon f \sinh \left( \frac{1}{2} \varepsilon h \right). \]

Using the above equation we can write the flow rule and yield condition as follows

\[ \frac{\partial v_x}{\partial x} = \lambda (2S_{xx} + S_{yy}), \]

\[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 3\lambda (S_{xx} + S_{yy}), \]

\[ \frac{\partial v_x}{\partial y} + \varepsilon^2 \frac{\partial v_y}{\partial x} = 2\varepsilon^2 \lambda \sigma_{xy}, \]

FIG. 6. Longitudinal and transverse variations of the second order velocity components at \( y = h(x) \) and \( \hat{x} = L/2 \), respectively, for different values of \( \hat{\mu}/\varepsilon \).
and
\[ 3(S_{xx}^2 + S_{yy}^2 + S_{xx}S_{yy} + \varepsilon^2 \sigma_{xy}^2) + 2\varepsilon f \cosh(\frac{1}{2} \sigma_{xy}) - (1 + \varepsilon^2 f^2) = 0. \] (85)

Under steady state conditions, Eqn (77) that determines the evolution of porosity can be written as
\[ \varepsilon \frac{\partial f}{\partial x} v_x + \varepsilon \frac{\partial f}{\partial y} v_y = (1 - \varepsilon f) \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right). \] (86)

The boundary conditions on the die–metal interface \( y = h(x) \) are given as
\[ v_y = h' v_x, \]
\[ \sigma_{xy} = \mu \sigma_{yy} + \mu' \sigma_{xx} - 2 \sigma_{xy} + \varepsilon^2 h' \sigma_{xy}. \] (88)

The constant flux and zero-exit-force boundary conditions are written as
\[ \int_{0}^{h(x)} [1 - \varepsilon f(x, y)] v_x(x, y) \, dy = h_0(1 - \varepsilon f_0), \] (89)
and
\[ \int_{0}^{h_1} \sigma_{xx}(1, y) \, dy = 0. \] (90)

Also, the entrance porosity condition is given as
\[ f(0, y) = \varepsilon_0. \] (91)

Equations (79)-(91) show that the solution of the normalized problem is of the form
\[ v = v(x, y, \mu, \varepsilon_0, \varepsilon), \]
\[ \sigma = \sigma(x, y, \mu, \varepsilon_0, \varepsilon), \]
\[ f = f(x, y, \mu, \varepsilon_0, \varepsilon), \]
and
\[ \lambda = \lambda(x, y, \mu, \varepsilon_0, \varepsilon). \]

3.3. Perturbation expansion
We attempt a perturbation expansion of the solution in the form
\[ v = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + O(\varepsilon^3), \]
\[ \sigma = \sigma^{(0)} + \varepsilon \sigma^{(1)} + \varepsilon^2 \sigma^{(2)} + O(\varepsilon^3), \]
\[ f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + O(\varepsilon^3), \]
\[ \lambda = \lambda^{(0)} + \varepsilon \lambda^{(1)} + \varepsilon^2 \lambda^{(2)} + O(\varepsilon^3). \]

Substituting the above expansions into the Eqns (79)-(91), we obtain the following hierarchy of problems. The leading order problem is identical to that of the fully dense metal [Eqns (32)-(41)].

At \( O(\varepsilon) \) the problem is given by
\[ \frac{\partial \sigma_{xx}^{(1)}}{\partial x} + \frac{\partial \sigma_{yy}^{(1)}}{\partial y} = 0, \]
\[ \frac{\partial \sigma_{xy}^{(1)}}{\partial y} = 0, \]
\[ \frac{\partial v_x^{(1)}}{\partial x} = \lambda^{(0)}(2S_{xx}^{(1)} + S_{yy}^{(1)}) + \lambda^{(1)}(2S_{xx}^{(0)} + S_{yy}^{(0)}), \]
\[ \frac{\partial v_x^{(1)}}{\partial x} + \frac{\partial v_y^{(1)}}{\partial y} = 3\lambda^{(0)}(S_{xx}^{(1)} + S_{yy}^{(1)}), \]
\[
\frac{\partial v^{(1)}_x}{\partial y} = 0, \quad (104)
\]

\[
3\left(2S^{(0)}_{xx}S^{(1)}_{xx} + 2S^{(0)}_{yy}S^{(1)}_{yy} + S^{(1)}_{xx}S^{(0)}_{yy} + S^{(0)}_{xx}S^{(1)}_{yy}\right) + 2f^{(0)}\cosh(\frac{1}{2} \sigma^{(0)}_{kk}) = 0, \quad (105)
\]

\[
S^{(1)}_{xx} + S^{(1)}_{yy} = \frac{1}{3} f^{(0)} \sinh(\frac{1}{2} \sigma^{(0)}_{kk}), \quad (106)
\]

\[
\frac{\partial f^{(0)}}{\partial x} v^{(0)}_x + \frac{\partial f^{(0)}}{\partial y} v^{(0)}_y = \frac{\partial v^{(1)}_x}{\partial x} + \frac{\partial v^{(1)}_y}{\partial y}. \quad (107)
\]

The boundary conditions to this order are as follows

\[
\sigma_{xy}^{(1)} = \mu \sigma_{yy}^{(1)} + h(S_{xx}^{(1)} - S_{yy}^{(1)}) \quad \text{on } y = h(x). \quad (108)
\]

\[
\int_0^{h(x)} \left[ v^{(1)}_x(x, y) - f^{(0)}(x, y)v^{(0)}_x(x) \right] dy = -h_0 f_0, \quad (109)
\]

\[
\int_0^{h(x)} \sigma_{xy}^{(1)}(1, y) dy = 0, \quad (110)
\]

and

\[
f^{(0)}(0, y) = f_0. \quad (111)
\]

In the following we also list the \( O(\varepsilon^2) \) equations that can be used for the determination of \( f^{(1)} \):

\[
\frac{\partial f^{(1)}}{\partial x} v^{(0)}_x + \frac{\partial f^{(1)}}{\partial y} v^{(0)}_y = \left( \frac{\partial f^{(2)}}{\partial x} v^{(2)}_x + \frac{\partial f^{(2)}}{\partial y} v^{(2)}_y \right) - f^{(0)} \left( \frac{\partial v^{(1)}_x}{\partial x} + \frac{\partial v^{(1)}_y}{\partial y} \right) - \left( \frac{\partial f^{(0)}}{\partial x} v^{(1)}_x + \frac{\partial f^{(0)}}{\partial y} v^{(1)}_y \right), \quad (112)
\]

\[
\frac{\partial v^{(2)}_x}{\partial x} + \frac{\partial v^{(2)}_y}{\partial y} = 3f^{(0)}(S^{(2)}_{xx} + S^{(2)}_{yy}) + 3f^{(1)}(S^{(1)}_{xx} + S^{(1)}_{yy}), \quad (113)
\]

\[
S^{(2)}_{xx} + S^{(2)}_{yy} = \frac{1}{3} f^{(1)} \sinh(\frac{1}{2} \sigma^{(0)}_{kk}) + \frac{1}{6} f^{(0)} \sigma^{(1)}_{kk} \cosh(\frac{1}{2} \sigma^{(0)}_{kk}). \quad (114)
\]

The boundary condition to be used in integrating Eqn (112) is

\[
f^{(1)}(0, y) = 0. \quad (115)
\]

3.4. Solution to the problem

The solution of the leading order problem is the same as that of the fully dense metal given in Section 2.3. The solution of the \( O(\varepsilon) \) problem is outlined in the following.

Rewriting Eqn (107), we have the following linear first-order partial differential equation for \( f^{(0)} \):

\[
\frac{\partial f^{(0)}}{\partial x} v^{(0)}_x + \frac{\partial f^{(0)}}{\partial y} v^{(0)}_y = \lambda^{(0)} f^{(0)} \sinh(\frac{1}{2} \sigma^{(0)}_{kk}). \quad (116)
\]

It can be readily shown that the solution of the above equation is

\[
f^{(0)}(x) = f_0 \exp \left[ -\sqrt{3} \int_0^x \frac{h(\eta)}{h(\eta) \sinh(\frac{1}{2} \sigma^{(0)}_{kk}(\eta))} d\eta \right]. \quad (117)
\]

which shows that porosity is, to leading order, independent of \( y \). The above equation is identical to that derived by Durban and Mear [10] who based their calculations on a "radial" velocity field. It should be noted, however, that in most cases the contribution of the second term \( f^{(1)} \) in the porosity expansion (98) is not insignificant and should be taken into account. This becomes clear in Section 4, where comparisons with detailed finite element calculations are made.
After some lengthy, but otherwise straightforward, calculations the solution of the $O(e)$ velocity and stress components is found to be

$$v^{(1)}_x(x) = v^{(0)}_x(x) \left[ f^{(0)}(x) - f_0 \right]. \quad (118)$$

$$v^{(1)}_y(x, y) = v^{(0)}_x(x) \frac{h'(x)}{h(x)} y, \quad (119)$$

and

$$\sigma^{(1)}_{kk}(x) = \sigma^{(1)}_{yy}(x) + a(x), \quad (120)$$

$$\sigma^{(1)}_{yy}(x) = \left[ \int_x^{1} \left( a'(\xi) + \frac{h'(\xi)}{h(\xi)} a(\xi) \right) \exp \left( - \mu \int_x^{1} \frac{d\eta}{h(\eta)} \right) d\xi - a(1) \right] \exp \left( \mu \int_x^{1} \frac{d\eta}{h(\eta)} \right), \quad (121)$$

$$\sigma^{(1)}_{xy}(x, y) = \left[ \mu \sigma^{(1)}_{yy}(x) + h'(x) a(x) \right] \frac{y}{h(x)}, \quad (122)$$

where

$$a(x) = - \frac{2}{\sqrt{3}} f^{(0)}(x) \cosh \left[ \frac{1}{2} \sigma^{(0)}_{kk}(x) \right], \quad (123)$$

$$b(x) = \frac{1}{3} f^{(0)}(x) \sinh \left[ \frac{1}{2} \sigma^{(0)}_{kk}(x) \right]. \quad (124)$$

It should be noted that for a fully dense material ($f = f_0 = 0$) the above equations reduce to $\sigma^{(1)} = 0$ and $v^{(1)} = 0$, which agrees with the solution developed in Section 2.3, where the second term in the stress and velocity expressions are $O(e^2)$.

Finally, using Eqns (112)-(115) we find the second term in the porosity expansion to be

$$f^{(1)}(x) = \exp \left[ - \int_0^x F_1(\eta) d\eta \right] \int_0^x F_2(\xi) \exp \left[ \int_0^\xi F_1(\eta) d\eta \right] d\xi, \quad (125)$$

where

$$F_1(x) = - \frac{\lambda^{(0)}(x)}{v^{(0)}_x(x)} \sinh \left( \frac{1}{2} \sigma^{(0)}_{kk}(x) \right), \quad (126)$$

and

$$F_2(x) = \frac{\sqrt{3}}{2} \frac{\lambda^{(0)}(x)}{v^{(0)}_x(x)} \left[ 3b(x) \left( 3b(x) - a(x) - \frac{2}{\sqrt{3}} f^{(0)}(x) \right) - \frac{1}{2} a(x) \sigma^{(1)}_{kk}(x) \right]. \quad (127)$$

Figure 7 shows the variation of the leading as well as the next order porosity in the reduction region of a straight die with $r = \Delta h/h_0 = 0.25$ for different values of the coefficient
of friction \( \mu \). The next order stresses and velocities, for the same die geometry are shown in Figs 8 and 9, respectively. It should be noted that \( \sigma_{xy}^{(0)} \) and \( v_y^{(0)} \) as well as \( \sigma_{xy}^{(1)} \) and \( v_y^{(1)} \) are linear in \( y \). It is also interesting to note that as \( \mu \) increases the pressure in the billet increases as well, thus causing a higher densification of the metals as shown in Fig. 7.

A comparison of the developed asymptotic solutions with the results of detailed finite element calculations both for the fully dense and the porous metals is given in the following section.

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![Graph](attachment:image.png)

**FIG. 8.** Longitudinal variation of the second order stress components for different values of \( \mu/e \) (porous metal).
4. FINITE ELEMENT SOLUTIONS

The asymptotic solutions developed in the previous sections are now compared with the results of detailed finite element calculations both for a fully dense and porous metal. The ABAQUS general purpose finite element program is used for the computations [17].
In all cases analysed, the area reduction is \( r = \Delta h / h_0 = 0.25 \), and the length of the reduction region is \( L = 2.5 \Delta h \), so that \( \varepsilon = \Delta h / L = 0.1 \). The reduction area of the die is shaped in the form of a fifth order polynomial with zero slope and curvature at both ends. The coefficient of friction along the die–metal interface is assumed to vanish.

The analysis is carried out incrementally using an updated Lagrangian formulation and Newton's method is used to solve the discretized equilibrium equations.

4.1. Fully dense metals

Four-node isoparametric elements with \( 2 \times 2 \) Gauss integration and an independent interpolation for the dilatation rate are used in order to avoid artificial constraints on incompressible modes [18]. An elastic–perfectly-plastic model is used in the calculations. The constitutive parameters used in the computations are \( \bar{E} / \bar{\sigma}_0 = 300 \) and \( \nu = 0.499 \), where \( \bar{E} \) is Young's modulus and \( \nu \) is Poisson's ratio. A rigid smooth piston pressing against the rear face of the billet provided the driving force.

The results of the asymptotic solution presented in Section 2 show that, for \( \beta = 0 \) and \( \varepsilon = 0.1 \) which is the case here, the contribution of the second order term in the stress expansion is less than 10%. Therefore, the results of the finite element calculations are compared in the following with the predictions of the leading term only in the asymptotic expansion of the solution.

Figure 10 shows the variation of the calculated extrusion force with the applied piston displacement together with the deformed finite element mesh at the end of the calculation. The active plastic elements are shown dark. The asymptotic solution predicts a steady state

![Graphs showing longitudinal variations of stress and plastic strain](image-url)
extrusion force to leading order

\[
\frac{\hat{F}^{(0)}}{\hat{\sigma}_0 \hat{h}_0} = \frac{2}{\sqrt{3}} \ln \frac{\hat{h}_0}{\hat{h}_1} = 0.332,
\]

which agrees very well with the finite element value of \( \hat{F}/(\hat{\sigma}_0 \hat{h}_0) = 0.335 \).

Figure 11 shows the variation of the hydrostatic stress \( \hat{S}_{kk}/(3\hat{\sigma}_0) \), the deviatoric stress \( \hat{S}_{xx}/\hat{\sigma}_0 \), and the equivalent plastic strain \( \hat{\varepsilon}^p \), along the direction of extrusion inside the reduction region of the die. The equivalent plastic strain is defined as

\[
\hat{\varepsilon}^p = \int_0^1 (\frac{2}{3} D_{ij}^p D_{ij}^p)^{1/2} dt,
\]

where \( t \) is time, and \( D^p \) is the plastic part of the deformation rate tensor. In Fig. 11 and in all subsequent figures, triangles indicate the results of the finite element calculations on \( y = 0 \), circles on \( y = h(x)/2 \), and squares on \( y = h(x) \). The solid lines in Fig. 11 indicate the prediction of the leading term in asymptotic expansion of the solution. The analytical solution agrees well with the results of the finite element calculations.

We mention that all stress components of \( O(\hat{\sigma}_0) \), i.e. \( \hat{\sigma}_{xx}, \hat{\sigma}_{yy} \) and \( \hat{\sigma}_{zz} \), can be determined from the values of \( \hat{S}_{xx} \) and \( \hat{S}_{kk} \) shown in Fig. 11; the shear stress \( \hat{\sigma}_{xy} \) is of \( O(\hat{\sigma}_0^2) \) and does not play an important role in this case.

It should be noted that as the material enters the die, it deforms elastically first. When the stresses becomes high enough to cause yielding, plastic flow occurs. The stress distribution
in the elastically deforming part of the billet is discussed briefly in Appendix II; a more detailed discussion can be found in Ref. [19]. The results of the asymptotic solution shown in Fig. 11 (solid lines) do include such elastic effects at the die-entrance.

4.2. Porous metals

Four-node isoparametric elements with $2 \times 2$ Gauss integration are used in the calculations. It should be noted that Gurson's plasticity model is not included in the ABAQUS "material library". This code, however, provides a general interface so that the user may introduce his own constitutive model in a "user subroutine". The integration of the elastoplastic equations for Gurson's model and the computation of the corresponding "linearization moduli" are implemented in ABAQUS using the method presented by Aravas [20]. The constitutive parameters used in the calculations are $\dot{E}/\dot{\sigma}_0 = 300$, $\nu = 0.49$, $\dot{\gamma}_m = \dot{\gamma}_0 = \text{constant}$, and $\dot{f}_0 = 0.10$.

Figure 12 shows the variation of the calculated extrusion force with the applied piston displacement together with the deformation finite element mesh at the end of the calculation. The finite element steady state value of the extrusion force is $F/(\dot{\sigma}_0 h_0) = 0.296$ and agrees very well with the value of 0.290 which is the prediction of the two-term asymptotic expansion of the solution.

Figure 13 shows the variation of the hydrostatic stress $\dot{\sigma}_{kk}/(3\dot{\sigma}_0)$, and the deviatoric stresses $\dot{S}_{xx}/\dot{\sigma}_0$ and $\dot{S}_{yy}/\dot{\sigma}_0$ along the direction of extrusion inside the reduction region. Curve I in that figure is the prediction of the leading term in the expansion of the solution, whereas curve II is the sum of the first and the second terms in the expansion.

Figure 14 shows the variation of the microscopic equivalent plastic strain $\tilde{\varepsilon}^p$ and the porosity $\tilde{f}$ along the extrusion direction.

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**Fig. 13.** Longitudinal variations of $\dot{\sigma}_{kk}/(3\dot{\sigma}_0)$, $\dot{S}_{xx}/\dot{\sigma}_0$ and $\dot{S}_{yy}/\dot{\sigma}_0$ for porous metal. Curve I indicates the leading order asymptotic solutions and curve II is the sum of the first two terms in the expansion of the solution. FEM solutions are denoted by $\triangle$ on $y = 0$, $\bigcirc$ on $y = h(x)/2$ and $\square$ on $y = h(x)$. 
Asymptotic analysis of extrusion of porous metals

Fig. 14. Longitudinal variations of the microscopic equivalent plastic strain $\varepsilon^p$ and the porosity $\tilde{f}$ for porous metal. Curve I indicates the leading order asymptotic solutions and curve II is the sum of the first two terms in the expansion of the solution. FEM solutions are denoted by △ on $y = 0$, ○ on $y = h(x)/2$, and □ on $y = h(x)$.

Figures 13 and 14 show that when the effects of the second term in the expansion of the solution are taken into account, the predictions of the asymptotic solutions are improved and agree well with results of the finite element analysis. The results of the asymptotic solution shown in Figs 13 and 14 include the elastic effects at the die-entrance as described in Appendix II.

5. CONCLUSION

An asymptotic analysis of extrusion through moderately tapering dies has been developed for fully dense as well as porous metals. For the case of fully dense metals, it is shown that the leading order $[O(1)]$ solutions involves "slab flow", and that the next term in the expansion of the solution is $O(\varepsilon^2)$. In the case of porous metals the effects of porosity enter as an $O(\varepsilon)$ correction to the leading order solution of the fully dense metal. The results of the asymptotic solutions agree very well those of detailed finite element calculations.

The solutions presented in the paper are for elastic–perfectly-plastic materials. The effects of strain hardening can be easily incorporated in the analysis; such effects are discussed in some detail in Ref. [19].

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REFERENCES


**APPENDIX I: THE EFFECTS OF ELASTICITY**

We consider a homogeneous isotropic elastic-perfectly-plastic medium that yields according to the von Mises criterion with associated flow rule. For simplicity, the material is assumed to be incompressible, i.e. the elastic Poisson’s ratio \( \nu = 0.5 \). The form of the elastoplastic constitutive equations is

\[
\frac{1}{2} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_i} \right) = \dot{\varepsilon}_{ij} + \frac{3}{2E} \ddot{\varepsilon}_{ij},
\]

where \( E \) is Young’s modulus, and \( \varepsilon \) denotes the Jaumann or co-rotational derivative defined as

\[
\dot{\varepsilon}_{ij} = \ddot{\varepsilon}_{ij} + \frac{1}{2} \left( \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_i} \right),
\]

\( \ddot{\varepsilon}_{ij} \) being the spin tensor:

\[
\ddot{W}_{ij} = \dot{W}_{ij} - \dot{W}_{kl} \dot{W}_{ij}.
\]

The yield strain \( \varepsilon_0 = \sigma_0/\dot{E} \) varies in most metals from \( 10^{-3} \) to \( 10^{-2} \). We consider die geometries for which \( \varepsilon = O(10^{-1}) \) and define

\[
\varepsilon_0 = \varepsilon^2.
\]

where \( \varepsilon_0 \) is now \( O(1) \) or smaller.

Under steady-state plane-strain deformation the elastoplastic constitutive equations can be written as

\[
\frac{\partial \varepsilon_x}{\partial x} = 2S_{xx} + \varepsilon^2 \varepsilon_0 \left[ \frac{\partial S_{xx}}{\partial x} + \frac{\partial S_{xx}}{\partial y} \frac{\partial \sigma_y}{\partial y} + \left( \frac{\partial \sigma_y}{\partial y} - \varepsilon^2 \frac{\partial \sigma_y}{\partial x} \right) S_{xx} \right],
\]

\[
\frac{\partial \varepsilon_y}{\partial y} = 2S_{yy} + \varepsilon^2 \varepsilon_0 \left[ \frac{\partial S_{yy}}{\partial y} + \frac{\partial S_{yy}}{\partial x} \frac{\partial \sigma_x}{\partial x} + \left( \frac{\partial \sigma_x}{\partial x} - \varepsilon^2 \frac{\partial \sigma_x}{\partial y} \right) S_{yy} \right],
\]

and

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0,
\]

where the dimensionless quantities defined in Eqsns (12)-(15) have been used.

The non-dimensional boundary value problem consists now of the equilibrium equations (16) and (17), the constitutive equations (A5)-(A7), the yield conditions (21), and the boundary conditions (22)-(25). Using Eqs (A5)-(A7) and taking into account that \( \delta S_{xx}/\delta x = \delta S_{yy}/\delta y = \delta S_{yy}/\delta x = 0 \) we readily conclude that the effects of elasticity enter the solutions to \( O(\varepsilon^4) \) or higher. It should also be noted that for \( \nu \neq 0.5 \) the effects of elasticity enter the solution to \( O(\varepsilon^2) \) or higher.

The conclusions drawn above, also hold good for the case of the porous metals discussed in Section 3.
APPENDIX II: ELASTIC EFFECTS AT THE ENTRANCE

As the billet enters the reduction region of die, it deforms elastically first. When the stresses become high enough to cause yielding, plastic flow occurs. In this Appendix we analyse briefly the deformation of the billet in the elastic region. For simplicity, we consider an incompressible linear elastic material. Under the assumption of small elastic strains the constitutive equations can be written as

\[
\frac{1}{2} \left( \frac{\partial \delta_i}{\partial x_j} + \frac{\partial \delta_j}{\partial x_i} \right) = \frac{3}{2E} S_{ij},
\]

where \( E \) is the Young's modulus. Using the dimensionless quantities defined in Eqns (12)-(15) and a procedure similar to that used in Section 2 we find that

\[
S_{ij} = S^{(0)}_{ij} + O(\varepsilon^2),
\]

and

\[
p = p^{(0)} + \varepsilon p^{(1)} + O(\varepsilon^2),
\]

where \( p = \delta_{11}/(3\delta_0) \) is the dimensionless hydrostatic stress.

\[
S^{(0)}_{xx} = -S^{(0)}_{yy} = -\frac{2}{3\delta_0} \frac{h(x)}{h_0},
\]

\[
\sigma^{(0)}_{xy}(x, y) = \left(2 - \frac{h'(x)}{3\delta_0} - p^{(0)}y\right)y,
\]

\[
p^{(0)}(x) = \left[ \int_0^x q(\xi) \exp \left( \mu \int_0^\xi \frac{d\eta}{h(\eta)} \right) d\xi + \rho_0 \right] \exp \left( -\mu \int_0^x \frac{d\eta}{h(\eta)} \right),
\]

\[
q(x) = \frac{2}{3\delta_0} \frac{h'(x)}{h(x)} \frac{S^{(0)}_{xx}(x)}{h(x)} \left[ \mu - 2h'(x) \right],
\]

\[
p^{(1)} = p_1 = \text{constant},
\]

and \( \rho_0 \) in Eqn (B6) is a constant.

In deriving the above equations, the assumption is made that the whole cross-section \([x = \text{const.}, 0 \leq y \leq h(x)]\) behaves elastically. Similarly, the results presented in Sections 2 and 3 of the paper are based on the assumption that the whole cross-section is yielding. The details of the solution in the intermediate region, where only part of the cross-section is yielding, is beyond the scope of this paper; a discussion of the solution in that region can be found in Ref. [19]. Here, we simply approximate the elastic-plastic boundary by a straight line \( x = x^{ep} \). In our calculations, the value of \( x^{ep} \) and the constants \( p_0 \) and \( p_1 \) are determined from the yield condition and the fact that the stress components \( \sigma_{xx} \) and \( \sigma_{xy} \) must be continuous at \( x = x^{ep} \).