# Backorder Penalty Cost Coefficient " $b$ ": What Could it Be? 

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The classical economic order quantity (EOQ) model with planned penalized backorders $(\mathrm{PB})$ relies on postulating a value for the backorder penalty cost coefficient, $b$, which is supposed to reflect the intangible adverse effect of the future loss of customer goodwill following a stockout. Recognizing that the effect of the future loss of customer goodwill should be not a direct penalty cost but a change in future demand, Schwartz (1966) modified the classical EOQ-PB model by eliminating the backorder penalty cost term from the objective function and assuming that the long-run demand rate is a decreasing, strictly convex function of the customer's "disappointment factor" (defined as the complement of the demand fill rate) following a stockout, which in turn is an increasing, strictly convex function of the demand fill rate. He called the new model a perturbed demand (PD) model. Schwartz provided convincing justification for his PD model and presented several variations of it in a follow-up paper, but he did not solve any of these models. In this paper, we solve Schwartz's original PD model and its variations, and we discuss the implications of their solutions, thus filling a gap in the literature left by Schwartz. Moreover, having been convinced that Schwartz's approach is more valid than the classical approach for representing the effect of the loss of customer goodwill following a stockout, but also recognizing that the classical approach is far more popular than the PD approach, because of its simplicity and because of tradition, we use the solution of the PD model to infer the value of $b$ in the classical model, thus providing one possible answer to the question, what could $b$ be? A noteworthy implication of the solution of Schwartz's original PD model is that the optimal fill rate is always 0 or 1 , rendering the inferred value of $b$ in the classical model 0 or $\infty$, respectively. Suspecting that the property of the PD function which is most likely responsible for producing this "bang-bang" type of result is strict convexity, we show that for the case where the PD function is proportional to an integer power, say $n$, of the fill rate, the optimal fill rate is always 0 or 1 , if and only if $n>1$, in which case the PD function is strictly convex in the fill rate.

Keywords: Economic Order Quantity; Stockout; Backorders; Perturbed Demand

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## 1. Introduction

Anyone who has taken or taught a course in inventory management is likely to have pondered at how to quantify the cost incurred by a stockout. A stockout may incur an immediate, direct cost to the firm, as well as a future, indirect cost. The direct cost depends on whether the unfilled demand is backordered and eventually fulfilled with a delay, or is cancelled. In the first case, the direct cost is related to the delayed delivery and may include extra administration costs, material handling and transportation costs for expediting the backordered items, fixed or variable contractual penalties, the loss of profit from selling the backordered items at a discounted price, the interest on the profit tied up in the backorder, etc. In the second case, the direct cost is the lost profit of the cancelled demand. In many practical situations, part of an unfilled demand is backordered and part of it is cancelled. In most cases, the direct cost may be calculated with some effort. The indirect cost is much harder to evaluate. It is related to the loss of customer goodwill due to the stockout, which may lead to a temporary or permanent decline in future demand and market share, especially in a competitive market environment.

The quantification of the indirect cost of stockouts has long been an unsatisfactorily resolved issue in the literature. The difficulty in determining an appropriate penalty rate for the indirect cost of stockouts has prompted many researchers to replace this rate by a constraint on the customer service level. For example, Çentikaya and Parlar (1998) take this approach for the economic order quantity (EOQ) model with planned backorders which is at the center of our study in this paper too. This approach may seem more appealing to practitioners, but it only transposes the problem of estimating an appropriate penalty rate for stockouts to one of determining an appropriate customer service level.

The EOQ model with planned backorders is one of the earliest models in inventory theory that deals with stockouts. It relies on postulating a value for the backorder penalty cost coefficient, denoted by $b$, which is supposed to reflect the intangible adverse effect of the loss of customer goodwill following a stockout. We refer to this model as the penalized backorders (henceforth, PB) model. The PB model is based on the following assumptions. A firm buys a single type of items from a supplier, holds them in inventory, and sells them to its customers upon demand. The demand for items, denoted by $D$, is continuous and constant over time, procurement and delivery of the items are instantaneous, and unfilled demand is backordered. Finally, the gross profit (selling price minus purchase price) per item sold, denoted by $p$, the fixed order cost, denoted by $k$, the inventory holding cost per item per unit time, denoted by $h$, and the backorder penalty cost per item per unit
time, denoted by $b$, are known and constant over time. The decision variables are the order quantity, denoted by $Q$, and the fraction of demand that is met from stock, known as fill rate, denoted by $F$.

All the parameters of the PB model, except $b$, may be more or less specified. Schwartz (1966) was one of the first to note that the effect of the loss of goodwill should not be a direct penalty cost of the type considered in the PB model, because the effect of goodwill loss is incurred not at the time of the stockout incident, but at a later time, due to the customer's disappointment caused by the stockout and his subsequent decision to lower his future demand. With this in mind, Schwartz (1966) modified the PB model by eliminating the explicit backorder penalty cost term from the objective function and assuming that the long-run demand rate - and hence the long-run average reward of the firm - is a function of the customer's "disappointment factor", which he defined as the fraction of demand not met from stock. Schwartz called the resulting model a perturbed demand (henceforth, PD) model.

To derive an analytical form of the perturbed demand as a function of the disappointment factor, Schwartz (1966) assumed the following customer response to stockouts. When a customer places an order and finds out that it cannot be delivered, he changes his a priori ordering pattern in the future by reducing the amount he would otherwise have bought in each of a number of future periods. The total amount that the customer does not buy because of the disappointment, denoted by $B$, and the maximum potential demand rate in a cycle with no disappointments, denoted by $A$, are finite. The above assumed customer response led to the following strictly convex long-run PD function:

$$
\begin{equation*}
D^{\prime}\left(F^{\prime}\right)=\frac{A}{1+\left(1-F^{\prime}\right) B}, \tag{1}
\end{equation*}
$$

where $F^{\prime}$ is the long-run average fill rate, and hence $1-F^{\prime}$ is the fraction of demand not met from stock, i.e., Schwartz's disappointment factor. We note that throughout this paper, we shall be using the notation $X^{\prime}$ for variables and functions in the PD model whose equivalent variable/function in the PB model is denoted by $X$, to distinguish between the two models.

The PD model proposed by Schwartz (1966) replaces the indeterminable task of subjectively choosing $b$ in the PB model with the better defined task of estimating parameters $A$ and $B$ of the PD function, $D^{\prime}\left(F^{\prime}\right)$. Schwartz (1966) proposed a procedure for measuring parameters $A$ and $B$ from observed demand data. This procedure is based on the assumption that when a customer faces a stockout, he reduces the size of his next order by some amount, the following one by a smaller amount, the next by a still smaller, and so on, so that as time passes, he tends to forget about the disappointment; therefore, his subsequent orders will approach their original level, $A$.

Schwartz (1966) provided convincing justification for his PD model, but he did not solve it. In a follow-up paper, Schwartz (1970) continued his investigation of the PD model by formulating three
different variations of it in which he replaced the explicit fixed order cost with a constraint on the order quantity, the interorder time, and the starting inventory in each cycle, respectively. For each variation he considered both cases with backlogging and lost sales. In all variations, he merely stated in a few lines the first-order condition for the optimal quantity of unfilled demand, but in none of these variations did he solve this condition or provide any further analysis, discussion, or insight. In this paper, we solve exactly the original PD model introduced by Schwartz (1966) and its three variations considered in Schwartz (1970), in the case of backlogging, thus filling a gap in the literature left by Schwartz. Moreover, we discuss the implications of the solutions.

In the last sentence of his conclusions, Schwartz (1970) wrote, "The Perturbed Demand approach to goodwill stockout penalties is both substantially more valid and more practical than any previously considered in the literature of inventory theory". We agree with the position that the PD approach to goodwill stockout penalties is in general more valid than the classical PB approach, although we must point out that the PD function (1) proposed by Schwartz (1966) is based on a specific consumer response assumption and is therefore one of many possible alternative functions. The main reason we agree with Schwartz's approach is that this approach bypasses the difficulty of defining the problem of how to choose a good - let alone the best - value for the backorder penalty cost coefficient (or the equivalent customer service level) in the classical approach. Another reason is that the classical approach has the following paradox embedded in it. It assumes that there is a backorder penalty cost which is supposed to reflect the future loss of demand due to the loss of customer goodwill following stockouts, yet at the same time it assumes that the demand is constant.

While the PD approach introduced by Schwartz $(1966,1970)$ spawned several follow-up papers, to date, the classical PB approach is still predominant in the inventory management research literature and in practice. One possible explanation for this predominance is tradition and the fact that many ERP systems and other decision support systems used in practice rely on the input of user-defined safety stocks or equivalent customer service levels which imply specific backorder/stockout cost coefficients. Another possible explanation is that, while the PD approach is more valid than the PB approach, it is more complicated and hence less appealing to practitioners than the classical approach.

Having been convinced that the PD model is more valid than the PB model but also recognizing that the PB model is more appealing and widely used than the PD model because of its simplicity and because of tradition, we use the solution of the PD model to infer the value of $b$ in the equivalent PB model, thus providing one possible answer to the question, what could $b$ be in the PB model? The way we infer $b$ is by setting the optimal decision variables, $Q^{,^{*}}$ and $F^{*^{*}}$, in the PD model, equal to the respective variables, $Q^{*}$ and $F^{*}$ (which are functions of $b$ ), in the PB model, and
solving for $b$. Once this is done, the resulting demand rate $D^{\prime}\left(F^{\prime *}\right)$ in the PD model is in general different than the constant demand rate $D$ in the PB model. This difference, however, can be justified if one thinks of $D$ as a "short-run" constant demand rate and $D^{\prime}\left(F^{\prime *}\right)$ as a "long-run" constant demand rate. The idea here is that as time passes, if one uses the correct decision variables based on the correct value of $b$ in the PB model, the short-run demand rate $D$ will drift towards $D^{\prime}$ $\left(F^{\prime *}\right)$, assuming that the fill rate $F^{\prime^{*}}$ is kept constant, so that in the long run, it will settle to $D^{\prime}\left(F^{\prime *}\right)$.

A noteworthy implication of the solution of Schwartz's original PD model is that the optimal fill rate is always 0 or 1 , making the inferred value of $b$ in the classical model 0 or $\infty$. Suspecting that the property of the PD function which is most likely responsible for producing this "bang-bang" type of result is strict convexity, we show that for the case where the PD function is proportional to an integer power, say $n$, of the fill rate, the optimal fill rate is always 0 or 1 , if and only if $n>1$, in which case the PD function is strictly convex in the fill rate. If $n=1$, in which case the PD function is not strictly convex in the fill rate, the bang-bang result does not hold. This reinforces our suspicion that the property of $D^{\prime}\left(F^{\prime}\right)$ which is most likely responsible for producing this bang-bang type of result is strict convexity.

We recognize that the PB model is a bit "tactical" relative to current inventory research. It is only an approximation to the stochastic ( $Q, r$ ) inventory model with backorders. As Çentikaya and Parlar (1998) point out, the relationship between the two models is analogous to the relationship between two classical inventory/production models, namely the deterministic multi-period model with backorders (Zangwill, 1969) and the stochastic multi-period model with backorders for which an ( $s, S$ ) policy is optimal (Scarf, 1960). The simple PD model in this paper and its computable results in terms of inferred backorder costs and decision variables may provide insight into the analysis of the $(Q, r)$ model with stochastic demands.

The rest of this paper is organized as follows. In Section 2 we review some of the relevant literature on the effect of stockouts. In Section 3, we summarize some more or less known results on the optimal decision variables of the classical PB model and three variations of it which correspond to the three variations described by Schwartz (1970), for the case of backlogging. In Section 4, we derive analytical expressions for the optimal decision variables of the respective PD models and the inferred value of $b$ in the PB models. In Section 5, we explore the role of the convexity of the PD function on the optimal fill rate of the PD model. Finally, we draw our conclusions in Section 6.

## 2. Literature Review

The effect of stockouts on current sales and future demand has been studied by the Operations Management (OM) community as well as by the Logistics Research (LR) community.

Some of the related work reported in the OM literature has been based on developing decision trees to model the consequences of stockouts (e.g., Chang and Niland, 1967) and using surveys to estimate the parameters of the trees (e.g., Oral et al., 1972, and Oral, 1981). Most of the research on the effects of stockouts on current and future sales in the OM literature, however, has focused on developing mathematical inventory control models in which demand is presumed to be a function of a certain direct or indirect quantitative measure of stockouts, such as fill rate, average delivery delay, etc. Examples of such work are Hanssmann (1959), Schwartz, (1966, 1970), Hill (1976), Caine and Plaut (1976), Robinson (1988), and Argon et al. (2001). The work in this paper follows this stream of research and in particular Schwartz $(1966,1970)$. There has also been a closely related stream of research in which demand is presumed to be a function of inventories. Examples of such work are Urban (1995) and Balakrishnan et al. (2004).

Schwartz's work and the works that followed it have remained within the framework of a single decision maker formulation and hence have not looked into the underlying competition interactions between suppliers. Given that the future defection of a customer depends on what other options he has, several researchers have addressed service-related issues within a game theoretic framework. There is a large body of OM literature that has looked at product and/or supplier substitution or switching when stockouts occur. Examples of such work are Li (1992), Ernst and Cohen (1992), Ernst and Powell (1995, 1998), Lippman and McCardle (1997), Netessine et al. (2006), Bernstein and Federgruen (2004a, b), and Dana and Petruzzi (2001). In all of the above works, the two factors - competition in product availability and its future effect - have been studied more or less separately. To the best of our knowledge, the only exceptions that have assumed that customer demand is a function of previous service encounters, are Gans (2002), Hall and Porteus (2000), Gaur and Park (2007), Liberopoulos and Tsikis (2007), Liu et al. (2007), and Olsen and Parker (2008).

Most of the work on the effects of stockouts reported in the LR literature has focused on identifying and explaining consumer reaction to stockouts in retail settings. Such reaction may include item (brand or variety) or purchase quantity switching, cancellation or deferral of purchase, store switching, etc. A number of studies have relied on postulating some decision model with alternative possible outcomes and courses of action of consumers and retailers following a stockout and estimating the parameters (probabilities, costs, etc.) of that model via interviews and/or mail
surveys. Examples of survey-based studies include Nielsen (1968a, b), Walter and Grabner (1975), Shycon and Sprague (1975), Schary and Becker (1978), Schary and Christopher (1979), Zinszer and Lesser (1981), Emmelhainz et al. (1991), Zinn and Liu (2001), Campo et al. (2000, 2004), and van Woensel et al. (2007). Two exceptions that focus on B2B rather than B2C markets are Dion et al. (1991) and Dion and Banting (1995). Another group of studies have been based on laboratory experiments. Examples are Charlton and Ehrenberg (1976), Motes and Castleberry (1985), and Fitzsimons (2000).

The above works have focused mainly on the immediate impact of stockouts on purchase incidence and choice decisions and not on the cumulative effects of stockouts over time. There are some studies that have looked at how stockouts affect future long-term demand of retailers, based on historical behavioral data analysis. Examples of such studies are Straughn (1991), Campo et al. (2003), Liberopoulos and Tsikis (2008), and Anderson et al. (2006). Finally, there exist some relatively recent survey- and experiment-based studies on consumers' perceptions of and reactions to waiting and service. Some examples are Taylor (1994), Carmon, et al. (1995), Hui and Tse (1996), Kumar et al. (1997), Zhou and Soman (2003), and Munichor and Rafaeli (2007).

It is worth noting that some of the empirical studies mentioned above provide supporting evidence that seems to validate Schwartz's assumption that when a customer places an order and finds out that it cannot be delivered, he changes his a priori ordering pattern in the future by reducing the amount he would otherwise have bought, but that as time passes, he tends to forget about the disappointment so that his subsequent orders would approach their original level. More specifically, Charlton and Ehrenberg (1976) conducted an experiment in which a panel of consumers was repeatedly offered the opportunity to buy certain artificial brands of a detergent, and showed that when a stockout condition was introduced and subsequently withdrawn, market shares and category sales returned to their pre-stockout levels with no apparent long-term effects. Motes and Castleberry (1985) repeated the same type of experiment using a real potato chip brand and also found that category sales returned to their pre-stockout levels. In another study, Schary and Becker (1978) reported the effects of a regional beer strike in which stockouts occurred in selected brands. Using brand share as the dependent variable, stockout effects were judged to be more short- than long-run. Dion and Banting (1995) reported the results of a study on the perceived consequences for business-to-business market buyers of being stocked out by their supplier and their repurchase loyalty on the next purchase occasion. The results showed that buyers often sought an alternate supplier in the face of a stockout, but the majority returned to the original supplier on the next purchase occasion. Zinn and Liu (2001) reported results of an interview-based study of consumer short-term response to stockouts. By comparing the perceptions of consumers who recently
experienced a stockout with those who did not, they showed that consumers appear able to isolate a recent stockout experience from their perception of other dimensions of the store's image. In another study, Campo et al. (2003) explored the impact of retail stockouts on whether, how much, and what to buy, by adjusting traditional purchase incidence, quantity and choice models, so as to account for stockout effects. Their study, which was based on scanner panel data of a large European supermarket chain, showed that stockouts may reduce the probability of purchase incidence and lead to the purchase of smaller quantities.

To the best of our knowledge, to date, there has been no empirical work aimed at estimating the size of the backorder (or any other stockout-related) cost coefficient. One exception is Badinelli (1986), who repeatedly asked decision makers to specify their marginal exchange rate between onhand inventory and backorders, and then used the relatively more exact inventory holding cost to estimate a "disvalue" cost function of the stockout performance measure through regression. In a somewhat related earlier work, Gardner and Dannenbring (1979) proposed that inventory decisions be seen as policy tradeoffs on a 3D response surface showing the optimal relationships among aggregate customer service (defined as the complement of the fill rate), workload (defined as the replenishment frequency) and investment, (defined as the sum of cycle and safety stock), regardless of the particular cost structure of the firm; however, they did not provide any information on how to obtain objective cost information.

## 3. The PB Model

In this section, we discuss the optimal decision variables in the classical PB model, namely, the order quantity, $Q$, and the fill rate, $F$. The only constraint on $Q$ is that it must be nonnegative. The fill rate must satisfy $0 \leq F \leq 1$. Note that if $F=0$, the firm operates in a pure make-to-order mode, backordering all the demand and not keeping any inventory. If $F=1$, on the other hand, the firm operates in a pure make-to-stock mode, keeping all items in inventory and not allowing any backorders. Finally, if $0<F<1$, the firm uses a mixed make-to-order and make-to-stock policy. We also derive and discuss the optimal decision variables for three variations of the PB model which are equivalent to the variations that Schwartz (1970) considered for the PD model. In these variations, the explicit fixed order cost is replaced with a constraint on the order quantity, the interorder time, and the starting inventory in each cycle, respectively.

The classical PB model and its three variations make up a total of four cases. For each case, it is straightforward to derive an expression of the average profit of the firm as a function of the decision variables $Q$ and $F$. Table 1 shows the average profit function, denoted by the $P(Q, F)$, and the
constraints for the four cases. The quantities $Q, Q / D$, and $Q F$ in the last column of Table 1 are the order quantity, the interorder time, and the starting inventory in each cycle, respectively, and $Q_{\text {min }}$, $T_{\min }$, and $I_{\min }$ are positive, finite numbers denoting the minimum values of these quantities, respectively. Parameters $Q_{\min }$ and $T_{\min }$ may be set either externally by the supplier, or internally by the firm to incur an implicit fixed order expense, if the explicit fixed order cost $k$ is not known or is difficult to obtain. Similarly, parameter $I_{\text {min }}$ may be set internally by the firm to incur an implicit fixed order expense, or as a safety stock against fluctuations in demand, because in reality demand may vary.

Table 1: Objective function and constraints for the classical PB model and its three variations

| Case | $P(Q, F)$ | Constraints |
| :---: | :---: | :---: |
| 1 | $p D-k \frac{D}{Q}-h \frac{Q F^{2}}{2}-b \frac{Q(1-F)^{2}}{2}$ | $0 \leq F \leq 1, Q \geq 0$ |
| 2 | $p D-h \frac{Q F^{2}}{2}-b \frac{Q(1-F)^{2}}{2}$ | $0 \leq F \leq 1, Q \geq Q_{\min }$ |
| 3 | $p D-h \frac{Q F^{2}}{2}-b \frac{Q(1-F)^{2}}{2}$ | $0 \leq F \leq 1, Q / D \geq T_{\text {min }}$ |
| 4 | $p D-h \frac{Q F^{2}}{2}-b \frac{Q(1-F)^{2}}{2}$ | $0 \leq F \leq 1, Q F \geq I_{\text {min }}$ |
|  |  |  |

Proposition 1 gives the optimal order quantity and fill rate that maximize the average profit subject to the constraints, for all four cases of the PB model, shown in Table 1.

Proposition 1: The optimal order quantity and average profit as a function of $F, Q^{*}(F)$ and $P^{*}(F)$, respectively, and the overall optimal order quantity, fill rate, and average profit, $F^{*}, Q^{*}$, and $P^{*}$, where $P^{*}=P\left(Q^{*}, F^{*}\right)$, for the classical PB model and its three variations shown in Table 1, are given in Table 2.

Table 2: Optimal decision variables and objective function for the classical PB model and its three variations

| Case | $Q^{*}(F)$ | $P^{*}(F)$ | $F^{*}$ | $Q^{*}$ | $P^{*}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $\sqrt{\frac{2 k D}{h F^{2}+b(1-F)^{2}}}$ | $p D-\sqrt{2 k D\left[h F^{2}+b(1-F)^{2}\right]}$ | $\frac{b}{h+b}$ | $\sqrt{\frac{2 k D(h+b)}{h b}}$ | $p D-\sqrt{\frac{2 k D h b}{(h+b)}}$ |
| 2 | $Q_{\min }$ | $p D-h \frac{Q_{\min } F^{2}}{2}-b \frac{Q_{\min }(1-F)^{2}}{2}$ | $\frac{b}{h+b}$ | $Q_{\min }$ | $p D-\frac{h b Q_{\min }}{2(h+b)}$ |


| 3 | $D T_{\min }$ | $p D-h \frac{D T_{\min } F^{2}}{2}-b \frac{D T_{\min }(1-F)^{2}}{2}$ | $\frac{b}{h+b}$ | $D T_{\min }$ | $p D-\frac{h b D T_{\min }}{2(h+b)}$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 4 | $\frac{I_{\min }}{F}$ | $p D-h \frac{I_{\min } F}{2}-b \frac{I_{\min }(1-F)^{2}}{2 F}$ | $\sqrt{\frac{b}{h+b}}$ | $I_{\min } \sqrt{\frac{h+b}{b}}$ | $p D-(\sqrt{b(h+b)}-b) I_{\min }$ |

The proof of Proposition 1 is trivial and is therefore omitted. It suffices to mention that the methodology to solve the optimization problems shown in Table 1 consists of the following four steps: (1) express the optimal order quantity as a function of $F$, say $Q^{*}(F)$, (2) write an expression for the average profit as a function of $F$ only, say $P^{*}(F)$, after having replaced $Q$ by $Q^{*}(F)$, i.e., $P^{*}$ $(F)=P\left(Q^{*}(F), F\right)$, (3) maximize $P^{*}(F)$, subject to $0 \leq F \leq 1$, to determine the optimal fill rate $F^{*}$, and (4) evaluate $Q^{*}\left(F^{*}\right)$ to determine the optimal order quantity $Q^{*}$. The implementation of these steps can be found in many textbooks on inventory management (e.g. Zipkin, 2000), at least for the classical PB model (case 1). For cases 2-4, it can be easily carried out in a similar manner.

Discussion of Proposition 1: From column 4 of Table 2, we can observe that in all four cases, the optimal fill rate, $F^{*}$, is a function of the backorder penalty cost coefficient, $b$. More specifically, in cases $1-3, F^{*}$ is given by the familiar newsvendor fraction, $b /(h+b)$, whereas in case 4 , it is given by the square route of this fraction. From Tables 1 and 2 , it is easy to see that if we set $Q_{\min }=D T_{\min }$, cases 2 and 3 are identical to each other. This means that there are really only three cases of the PB model to consider; however, we purposely leave the results for both cases 2 and 3 in Table 2, even though there are identical, because in Section 4 we will relate them to the results of the respective cases of the PD model, which are not identical. From Table 1, it is also easy to see that in cases 2-4, the average profit, $P(Q, F)$, is strictly decreasing in the order quantity $Q$ and that $Q$ is only restricted by a lower limit. For this reason, the optimal order quantity $Q^{*}$ is simply set at this lower limit, as can be seen in column 5 of Table 2. In cases 2 and 3, this limit is independent of $F$ and is equal to $Q_{\min }$ and $D T_{\min }$, respectively. In case 4 , it is given by $I_{\min } / F$, which becomes $I_{\min } / F^{*}$ once the optimal fill rate $F^{*}$ is specified. Case 1 is the only case where $P(Q, F)$ is not strictly decreasing in $Q$, because of the extra fixed order cost term, $-k D / Q$, which is increasing in $Q$. In this case, the optimal order quantity is given by the familiar square root formula in Table 2.

To summarize, in all cases, $F^{*}$ is a function of $b$. Moreover, in cases 1 and $4, Q^{*}$ is a function of $F^{*}$, and hence also a function of $b$. In cases 2 and 3 , on the other hand, $Q^{*}$ does not depend on $b$.

## 4. The PD Model

For each of the four variations of the PB model discussed in Section 3, we can construct an equivalent PD model. Table 3 shows the average profit function, denoted by the $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$, and the constraints for the four equivalent PD models, where $Q^{\prime}$ and $F^{\prime}$ are the decision variables. Case 1 is the original PD model introduced by Schwartz (1966) and cases 2-4 are the variations of the PD model considered in Schwartz (1970).

Table 3: Objective function and constraints for the original PD model and its three variations

| Case | $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ | Constraints |
| :---: | :---: | :---: |
| 1 | $p D^{\prime}\left(F^{\prime}\right)-k \frac{D^{\prime}\left(F^{\prime}\right)}{Q^{\prime}}-h \frac{Q^{\prime} F^{\prime 2}}{2}$ | $0 \leq F^{\prime} \leq 1, Q^{\prime} \geq 0$ |
| 2 | $p D^{\prime}\left(F^{\prime}\right)-h \frac{Q^{\prime} F^{\prime 2}}{2}$ | $0 \leq F^{\prime} \leq 1, Q^{\prime} \geq Q_{\text {min }}$ |
| 3 | $p D^{\prime}\left(F^{\prime}\right)-h \frac{Q^{\prime} F^{\prime 2}}{2}$ | $0 \leq F^{\prime} \leq 1, Q^{\prime} / D^{\prime}\left(F^{\prime}\right) \geq T_{\text {min }}$ |
| 4 | $p D^{\prime}\left(F^{\prime}\right)-h \frac{Q^{\prime} F^{\prime 2}}{2}$ | $0 \leq F^{\prime} \leq 1, Q^{\prime} F^{\prime} \geq I_{\text {min }}$ |

Proposition 2 gives the optimal order quantity and fill rate that maximize the average profit subject to the constraints, for all four cases of the PD model, shown in Table 3. It also gives the inferred backorder penalty cost coefficient, $b$, and the resulting optimal order quantity for the respective cases of the PB model, shown in Table 1.

Proposition 2: Suppose that $D^{\prime}\left(F^{\prime}\right)$ is given by (1). Then the optimal decision variables for the original PD model and its three variations shown in Table 3, along with the conditions under which they hold, as well as the inferred backorder penalty cost coefficient and resulting optimal order quantity for the respective PB models shown in Table 1, are given by Table 4, where

$$
\begin{gather*}
F_{2}^{\prime}=\text { smallest real root of }\left\{\frac{p A B}{\left[1+\left(1-F^{\prime}\right) B\right]^{2}}-h Q_{\min } F^{\prime}\right\},  \tag{2}\\
F_{3}^{\prime}=1+\frac{1}{B}-\sqrt{\frac{(1+B)^{2}}{B^{2}}-\frac{2 p}{h^{2} T_{\min }^{2}}} \tag{3}
\end{gather*}
$$

Table 4: Optimal decision variables and conditions under which they hold for the original PD model and its three variations, and inferred backorder penalty cost coefficient and resulting optimal order quantity for the respective PB models

| Case | PD Model |  |  | PB Model |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $F^{* *}$ | $Q^{\prime *}$ | Condition | $b$ | $Q^{*}$ |
| 1 | 0 | $\infty$ | $p \sqrt{A} B /(1+B)<\sqrt{2 h k}$ | 0 | $\infty$ |
|  | 1 | $\sqrt{\frac{2 k A}{h}}$ | $p \sqrt{A} B /(1+B)>\sqrt{2 h k}$ | $\infty$ | $\sqrt{\frac{2 k D}{h}}$ |
|  | 0,1 | $\infty, \sqrt{\frac{2 k A}{h}}$ | $p \sqrt{A} B /(1+B)=\sqrt{2 h k}$ | $0, \infty$ | $\infty, \sqrt{\frac{2 k D}{h}}$ |
| 2 | $F_{2}^{\prime}$ | $Q_{\text {min }}$ | $\begin{aligned} & p A B<h Q_{\min } \text { or } \\ & p A B=h Q_{\min }, B>0.5 \text { or } \end{aligned}$ | $h \frac{F_{2}^{\prime}}{1-F_{2}^{\prime}}$ | $Q_{\text {min }}$ |
|  | 1 | $Q_{\text {min }}$ | $\begin{aligned} & p A B>h Q_{\min }, B>0.5,{P^{\prime *}}^{*}\left(F_{2}^{\prime}\right)>p A-h Q_{\min } / 2 \\ & p A B \geq h Q_{\min }, B \leq 0.5 \text { or } \\ & p A B>h Q_{\min }, B>0.5, P^{\prime *}\left(F_{2}^{\prime}\right)<p A-h Q_{\min } / 2 \end{aligned}$ | $\infty$ | $Q_{\text {min }}$ |
|  | $F_{2}^{\prime}, 1$ | $Q_{\text {min }}$ | $p A B>h Q_{\text {min }}, B>0.5, P^{\prime *}\left(F_{2}^{\prime}\right)=p A-h Q_{\text {min }} / 2$ | $h \frac{F_{2}^{\prime}}{1-F_{2}^{\prime}} \infty$ | $Q_{\text {min }}$ |
| 3 | $F_{3}^{\prime}$ | $\frac{A T_{\min }}{1+\left(1-F_{3}^{\prime}\right) B}$ | $2 p B /(2+B)<h T_{\text {min }}$ | $h \frac{F_{3}^{\prime}}{1-F_{3}^{\prime}}$ | $D T_{\text {min }}$ |
|  | 1 | $A T_{\text {min }}$ | $2 p B /(2+B) \geq h T_{\text {min }}$ | $\infty$ | $D T_{\text {min }}$ |
| 4 | 0 | $\infty$ | $2 p A B /(1+B)<h I_{\text {min }}$ | 0 | $\infty$ |
|  | 1 | $I_{\text {min }}$ | $2 p A B /(1+B)>h I_{\text {min }}$ | $\infty$ | $I_{\text {min }}$ |
|  | 0,1 | $\infty, I_{\text {min }}$ | $2 p A B /(1+B)=h I_{\text {min }}$ | $0, \infty$ | $\infty, I_{\text {min }}$ |

The proof of Proposition 2 is given in the Appendix. Although the methodology to solve the optimization problems shown in Table 3 is standard and consists of the same four steps for solving the four cases of the PB model, outlined in Section 3, implementing this methodology on the PD model is not trivial, as is the case in the PB model, because the PD function $D^{\prime}\left(F^{\prime}\right)$ given by (1) significantly complicates the average profit function $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ shown in column 2 of Table 3.

Discussion of Proposition 2: From columns 2-4 of Table 4, we can see that the results for cases 1 and 4 of the PD model are similar to each other, and so are the results for cases 2 and 3 . This was also true for the respective cases of the PB model, discussed in Section 3.

The most striking similarity between cases 1 and 4 of the PD model is that in both cases the optimal fill rate $F^{* *}$ is always 0 or 1 . This means that in these two cases, it is always optimal to only hold inventory and not allow any backorders $\left(F^{*}=1\right)$, or to only allow backorders and not hold any inventory $\left(F^{\prime *}=0\right)$. Holding inventory and allowing backorders is never optimal.

On the other hand, in both cases 2 and 3 the optimal fill rate $F^{\prime *}$ is always 1 or equal to a quantity between 0 and 1 , even if the inventory holding cost rate, $h$, is extremely large, as long as it is not infinite. This quantity is denoted by $F_{2}^{\prime}$ and $F_{3}^{\prime}$ for cases 2 and 3 , respectively, and depends on the model parameters; therefore, it can assume a continuum of values. This means that in these two cases, it is always optimal either to only hold inventory and not allow any backorders ( $F^{*}=1$ ), or to allow backorders for some time and hold inventory for the rest of the time $\left(0<F^{*}<1\right)$.

The only interpretation that we can give about why in cases 1 and $4 F^{\prime *}$ can be 0 , whereas in cases 2 and 3 it assumes a continuum of finite nonzero values, is the following. In cases 1 and $4, F^{*}$ can be 0 , because $Q^{\prime *}$ can go to $\infty$, and when $Q^{,^{*}}$ goes to $\infty, F^{\prime *}$ must be 0 , as we will explain shortly. In cases 2 and 3 , on the other hand, $Q^{\prime *}$ cannot go to $\infty$, and therefore $F^{\prime *}$ does not have to be 0 (in fact, it cannot be 0 , because the first derivative of the average profit $P^{\prime^{*}}\left(F^{\prime}\right)$ is always positive at $F^{\prime}=0$ ). Note, however, that if we impose a finite upper limit, say $Q_{\max }$, on the order quantity, then it can be shown that in cases 1 and $4, F^{\prime *}$ cannot be 0 but instead assumes a continuum of positive values, just like in cases 2 and 3 . At the same time, if the minimum order quantity $Q_{\min }$ in case 2 , or the minimum interorder time $T_{\min }$ in case 3 , goes to $\infty$, then $Q^{,^{*}}$ will also go to $\infty$, and therefore $F^{\prime *}$ will go to 0 , just like in cases 1 and 4 .

In cases 1 and 4, the decisive condition of whether to only hold inventory $\left(F^{\prime *}=1\right)$ or only allow backorders $\left(F^{\prime *}=0\right)$ is $p \sqrt{A} B /(1+B)<\sqrt{2 h k}$ and $2 p A B /(1+B)<h I_{\text {min }}$, respectively. From these conditions, we can see that in both cases, increasing the reward and demand related parameters, $p$, $A$, or $B$, tends to favor the solution $F^{\prime *}=1$, i.e., only hold inventory. On the other hand, increasing the cost related parameters, $h$ or $k$, in case 1 , and similarly increasing $h$ or $I_{\min }$, in case 4 , tends to favor the solution $F^{\prime *}=0$, i.e., only allow backorders. In case 1 , the parameter that affects mostly the decisive condition is the price margin $p$, because it appears linearly in this condition, whereas parameters $h, k$ and $A$ appear in a square root, and parameter $B$ appears in a term that ranges between 0 and 1 . In case 4 , on the other hand, parameters $p, A, h$, and $I_{\text {min }}$ affect equally strongly the decisive condition, because they appear linearly in this condition. In contrast, the effect of parameter $B$ is weaker, because $B$ appears in a term that ranges between 0 and 1 .

From columns 2 and 3 of Table 4, we can see that in all cases, if $F^{\prime^{*}}=1$, then $Q^{\prime *}$ is finite. If $F^{\prime *}$ $=0$, however, which is true only in cases 1 and 4 , then $Q^{\prime^{*}}=\infty$, as we mentioned earlier. The reason for this is slightly different in each case. More specifically, in both cases, the appropriate decisive condition determines whether $F^{\prime^{*}}=0$ or 1 . The tradeoff at stake, favoring one or the other solution, is between incurring high inventory costs (and, in case 1 , high ordering costs as well) on one hand,
and losing long-term demand and therefore revenue, on the other hand. If the model parameters in the decisive condition are such that $F^{*^{*}}=0$, then it is optimal for the firm to operate strictly with planned backorders and no inventory. Since backorders incur no direct cost, the firm can have as many of them as it pleases for free. This much is true for both cases 1 and 4 . The difference in why $Q^{\prime *}=\infty$, between the two cases, is the following. In case 1 , given that the firm pays an order cost $k$ every time it orders a quantity $Q^{\prime}$, then why not have $Q^{\prime}$ be infinite to avoid paying the order cost? Hence, $Q^{\prime^{*}}=\infty$. In case 4 , on the other hand, if $F^{\prime^{*}}=0$, then $Q^{\prime^{*}}$ must be infinite, not to avoid paying the order cost, since there is no such cost, but because otherwise, the minimum-inventory constraint, $Q^{\prime} F^{\prime} \geq I_{\min }$, will be violated. Of course, in reality, the order quantity cannot be infinite. This can be handled in the model by assuming that the order quantity has an upper limit, say $Q_{\max }$, which is large enough so that $Q_{\max } \geq \sqrt{2 k A / h}$, in case 1 , and $Q_{\max } \geq I_{\min }$, in case 4 , and then resolving the optimization problem with the additional constraint $Q^{\prime} \leq Q_{\max }$ to obtain $F^{\prime^{*}}$. As was mentioned earlier, it can be shown that if we impose such a limit, $F^{,^{*}}$ cannot be 0 but instead assumes a continuum of positive values, just like in cases 2 and 3.

For the cases where $F^{\prime *}$ is always either one or between zero and one, i.e., cases 2 and 3 , the decisive condition of whether to only hold inventory $\left(F^{\prime *}=1\right)$ or allow backorders for some time and hold inventory for the rest of the time $\left(0<F^{\prime *}<0\right)$ is as follows. In case 2 , the decisive condition is complicated and can be analyzed into three levels of subconditions. These subconditions are $p A B<h Q_{\min }$, at the first level, $B>0.5$, at the second level, and $P^{\prime^{*}}\left(F_{2}^{\prime}\right)>p A-$ $h Q_{\min } / 2$, at the third level. More specifically, if $p A B<h Q_{\min }$, then the inventory holding cost is high enough compared to the loss of revenue caused by a drop in long-term demand, so that the firm can afford to allow some backorders, no matter how small $B$ is, as long as it is not zero; hence $F^{,^{*}}<1$. If $p A B \geq h Q_{\min }$, however, then the inventory holding cost may not be high enough compared to the loss of revenue caused by a drop in long-term demand to always allow some backorders. In this case, whether to allow some backorders or not depends on the value of $B$. Namely, if $B \leq 0.5$, the revenue term in $P^{\prime^{*}}(F)$ always increases faster than the inventory cost term, as $F^{\prime}$ increases from 0 to 1 , and therefore, $F^{\prime^{*}}=1$, i.e., no backorders are allowed (see Appendix). If $B>0.5$, on the other hand, the revenue term in $P^{\prime^{*}}\left(F^{\prime}\right)$ increases faster than the inventory cost term, as $F^{\prime}$ increases from 0 to the smallest root of the derivative of $P^{\prime^{*}}\left(F^{\prime}\right), F_{2}^{\prime}$, then the reverse is true as $F^{\prime}$ increases from $F_{2}^{\prime}$ to the second smallest root, and finally the revenue term increases faster than the inventory cost term again, as $F^{\prime}$ increases from the second smallest root to 1 . In this case, the optimal fill rate depends on whether $P^{\prime^{*}}\left(F_{2}^{\prime}\right)$ or $P^{\prime^{*}}(1)$ is larger.

From the first subcondition of case $2, p A B<h Q_{\min }$, we can see that increasing parameters $p, A$, or $B$, tends to favor the solution $F^{,^{*}}=1$, i.e., hold inventory and do not allow any backorders, whereas increasing $h$ or $Q_{\min }$ tends to favor the solution $F^{\prime^{*}}=F_{2}^{\prime}<1$, i.e., hold inventory but also allow some backorders. We can also see that all five parameters affect the first decisive condition equally strongly, because they all appear linearly in this condition. Finally, increasing the minimum order quantity $Q_{\min }$, decreases the smallest real root of expression (2) and hence $F_{2}^{\prime}$. In fact, as $Q_{\text {min }}$ tends to infinity, $F_{2}^{\prime}$ tends to zero. This is because, as $Q_{\min }$ tends to infinity, the firm is obliged to order a quantity that tends to infinity. If it keeps this quantity in stock, its inventory holding cost will also tend to infinity. To avoid this, it is preferable for the firm to backorder this quantity and pay the price of a reduced long-run demand rate, which is certainly finite.

In case 3, the decisive condition of whether to only hold inventory or allow backorders for some time and hold inventory for the rest of the time is $2 p B /(2+B)<h T_{\min }$. From this condition, we can see that, similarly to case 2 , increasing parameters $p$ or $B$, tends to favor the solution $F^{\prime^{*}}=1$, whereas increasing $h$ or $T_{\min }$ tends to favor the solution $F^{*^{*}}=F_{3}^{\prime}<1$. Moreover, parameters $p, h$, and $T_{\min }$ affect equally strongly the decisive condition, because they appear linearly in this condition, whereas the effect of parameter $B$ is weaker, because $B$ appears in a term that ranges between zero and one. Unlike, case 2, and for this matter cases 1 and 4 as well, in case 3 , the maximum potential demand rate $A$ is missing from the decisive condition as well as from the expression for $F_{3}^{\prime}$ given by (3). This is because $A$ appears linearly in all the terms of the average profit $P^{\prime^{*}}\left(F^{\prime}\right)$; therefore, all $A$ does is simply multiply $P^{\prime^{*}}\left(F^{\prime}\right)$ and its derivative by a constant without really affecting their roots. Also, it can be seen from (3) that increasing the minimum interorder time $T_{\min }$, decreases $F_{3}^{\prime}$. In fact, as $T_{\min }$ tends to infinity, $F_{3}^{\prime}$ tends to zero, essentially for the same reason that $F_{2}^{\prime}$ tends to zero, as $Q_{\min }$ tends to infinity, explained in the preceding paragraph.

Finally, recall from our discussion in Section 3, that cases 2 and 3 of the PB model are identical to each other, if $Q_{\min }=D T_{\min }$. This is no longer true for cases 2 and 3 of the PD model, because the demand rate is not a constant, as was the case in the PB model, but a function of the fill rate $F^{\prime}$.

From column 5 of Table 4, it can be seen that in cases 1 and 4, the inferred value of $b$ is 0 or $\infty$, because in these cases $F^{\prime *}$ is always equal to 0 or 1 , as was discussed earlier. In cases 2 and 3 , on the other hand, the inferred value of $b$ is either $\infty$ or equal to a finite number, because in these cases $F^{, *}$ is always 1 or equal to a number between 0 and 1 .

From column 3 of Table 4, it can be seen that in the subcase of case 1 of the PD model, where $p \sqrt{A} B /(1+B) \geq \sqrt{2 h k}$, as well as in case 3 , the optimal order quantity $Q^{\prime^{*}}$ is a function of $D^{\prime}\left(F^{*}\right)$. From the last column of the same table, it can also be seen that in the respective cases of the PB model, if we use the inferred value of $b$, shown in the second to last column of Table 4, then the resulting optimal order quantity $Q^{*}$ is given by the same function, but with $D^{\prime}\left(F^{*}\right)$ in the place of $D$. At a first glance, this seems to suggest that in these cases, the inferred value of $b$, which by definition guarantees that $F^{*}=F^{\prime^{*}}$, does not guarantee that $Q^{*}=Q^{\prime *}$. This further suggests that in these cases, there exist no two models - a PB and a respective PD model - with the same optimal decision parameters. This is true in the short run. As was mentioned in Section 1, however, if the firm uses the optimal parameters $F^{*}$ and $Q^{*}$ in the PB model, then as time passes, no matter what the initial value of $D$ is, the average demand rate will drift towards $D^{\prime}\left(F^{*}\right)$ which is equal to $D^{\prime}\left(F^{\prime *}\right)$, assuming that the fill rate $F^{*}$ is kept constant and equal to $F^{{ }^{*}}$, so that in the long run, its average value will be equal to $D^{\prime}\left(F^{\prime *}\right)$. Therefore, in the long run, the optimal order quantity $Q^{*}$ in the PB model will be equal to the optimal order quantity in the respective PD model.

## 5. On the Role of the Convexity of $\boldsymbol{D}^{\boldsymbol{\prime}}\left(\boldsymbol{F}^{\prime}\right)$ on $\boldsymbol{F}^{*}$

From Proposition 2, we saw that in Schwartz's original PD model (case 1), in which $D^{\prime}\left(F^{\prime}\right)$ is given by (1), the optimal fill rate, $F^{\prime^{*}}$, is 0 or 1 , rendering the inferred value of $b$ in the PB model equal to 0 or $\infty$, respectively. It is natural to suspect that the property of $D^{\prime}\left(F^{\prime}\right)$ which is most likely responsible for producing this "bang-bang" type of result is strict convexity; therefore, a question that arises logically is whether this bang-bang result holds for all strictly convex PD functions. Unfortunately, it is practically impossible to provide a clear answer to this question by analytical means. Instead, we can only provide a clue by means of the following propositions.

Proposition 3: Suppose that the PD function, $D^{\prime}\left(F^{\prime}\right)$, is positive, increasing, continuous and twice differentiable in $[0,1]$. Then, the following holds concerning the optimal fill rate, $F^{\prime *}$, in the original PD model shown in Table 3 (case 1):

1. If $\frac{D^{\prime}\left(F^{\prime}\right)}{F^{\prime 2}} \leq \frac{k h}{2 p^{2}}$, for all $F^{\prime} \in[0,1]$, then $F^{\prime^{*}}=0$.
2. If $\frac{D^{\prime}\left(F^{\prime}\right)}{F^{\prime 2}}>\frac{k h}{2 p^{2}}$, for some $F^{\prime} \in[0,1]$, and $\frac{d^{2} D^{\prime}\left(F^{\prime}\right)}{d F^{\prime 2}}>0$ and $\frac{d D^{\prime}\left(F^{\prime}\right)}{d F^{\prime}} \geq 4 \frac{D^{\prime}\left(F^{\prime}\right)}{F^{\prime}}$, for all $F^{\prime} \in[0,1]$, then $F^{*^{*}}$ is either 0 or 1 .

The proof of Proposition 3 is in the Appendix.

Discussion of Proposition 3: The first part of Proposition 3 states a sufficient condition that $D^{\prime}\left(F^{\prime}\right)$ must satisfy in order for the optimal fill rate to be 0 . If this condition is not met, then the second part states two other sufficient conditions that $D^{\prime}\left(F^{\prime}\right)$ must satisfy in order for the optimal fill rate to be either 0 or 1 . The first of these two conditions is that $D^{\prime}\left(F^{\prime}\right)$ must be strictly convex. The second condition is that $D^{\prime}\left(F^{\prime}\right)$ must satisfy the inequality $d D^{\prime}\left(F^{\prime}\right) / d F^{\prime} \geq 4 D^{\prime}\left(F^{\prime}\right) / F^{\prime}$. It is interesting to note that Schwartz's PD function given by (1) is strictly convex, and therefore satisfies the first condition, but it does not satisfy the second condition, which can be expressed as $B\left(5 F^{\prime}-4\right) \geq 4$. Yet, by Proposition 2 (case 1 ), $F^{\prime *}$ is still 0 or 1.

To help understand what the condition $d D^{\prime}\left(F^{\prime}\right) / d F^{\prime} \geq 4 D^{\prime}\left(F^{\prime}\right) / F^{\prime}$ means and how it is related to convexity, suppose that $D^{\prime}\left(F^{\prime}\right)$ is simply proportional to a positive power of $F^{\prime}$, namely,

$$
\begin{equation*}
D^{\prime}\left(F^{\prime}\right)=A F^{m}, \tag{4}
\end{equation*}
$$

where $A$ is a positive real number and $n$ is a nonnegative real integer. Then, the first condition (strict convexity) implies that $n>1$, whereas the second condition $\left(d D^{\prime}\left(F^{\prime}\right) / d F^{\prime}>4 D^{\prime}\left(F^{\prime}\right) / F^{\prime}\right)$ implies that $n>4$. Clearly, if the second condition is satisfied, i.e., if $n>4$, then the first condition is also satisfied, i.e., $n>1$; hence, by Proposition $3, F^{*}$ is 0 or 1 . In other words, the second condition is more restrictive than strict convexity.

What happens if $1<n \leq 4$, however? Then, the second condition no longer holds, but strict convexity holds. Does the bang-bang type of result still hold? The following proposition gives an answer to this question.

Proposition 4: Suppose that $D^{\prime}\left(F^{\prime}\right)$ is given by (4). Then, the optimal decision variables for the original PD model shown in Table 3 (case 1), along with the conditions under which they hold, are given in Table 5, for different values of $n$ :

Table 5: Optimal decision variables and conditions under which they hold for the original PD model in which the perturbed demand function is given by (4)

| n | PD Model |  |  |
| :---: | :---: | :---: | :---: |
| $F^{,^{*}}$ | $Q^{,^{*}}$ | Condition |  |
| 1 | $\frac{2 A p^{2}}{9 k h}$ | $3 \frac{k}{p}$ |  |
|  | $A p^{2}<\frac{9}{2} k h$ |  |  |
|  |  | $\sqrt{\frac{2 k A}{h}}$ |  |
|  |  | $A p^{2} \geq \frac{9}{2} k h$ |  |


| 2 | 0 | $\sqrt{\frac{2 k A}{h}}$ | $A p^{2}<2 k h$ |
| :---: | :---: | :---: | :---: |
| 1 | $\sqrt{\frac{2 k A}{h}}$ | $A p^{2}>2 k h$ |  |
|  | $[0,1]$ | $\sqrt{\frac{2 k A}{h}}$ | $A p^{2}=2 k h$ |
| $>2$ | 0 | 0 | $A p^{2}<2 k h$ |
| 1 | $\sqrt{\frac{2 k A}{h}}$ | $A p^{2}>2 k h$ |  |
|  | 0,1 | $0, \sqrt{\frac{2 k A}{h}}$ | $A p^{2}=2 k h$ |
|  |  |  |  |

The proof of Proposition 4 is in the Appendix.

Discussion of Proposition 4: Proposition 4 essentially states that if $D^{\prime}\left(F^{\prime}\right)$ is given by (4), then the bang-bang type of results holds if $D^{\prime}\left(F^{\prime}\right)$ is strictly convex and does not hold if $D^{\prime}\left(F^{\prime}\right)$ is not strictly convex. This result together with the result of Proposition 2, case 1, according to which the bangbang type result folds if $D^{\prime}\left(F^{\prime}\right)$ is given by (1), which is strictly convex, reinforces our suspicion that the property of $D^{\prime}\left(F^{\prime}\right)$ which is most likely responsible for producing this bang-bang type of result is strict convexity.

Of course, a more fundamental question is whether it is reasonable to assume that the PD function is strictly convex. Badinelli (1986) argues that in realistic cases, an inventory manager could reason that starting from a situation of excellent service (i.e., one with little or no backorders and low stockout risk) an incremental increase in stockouts would look bad. As the situation grows worse, subsequent increases might not look as serious as the first. Hence the manager's stockout cost as a function of the average backorders or stockout risk would exhibit a diminishing marginal cost which would yield a concave disvalue function. The analogy of this behavior, in the context of our model, is that the demand rate as a function of the disappointment factor $\left(1-F^{\prime}\right)$ would exhibit a diminishing marginal decrease which would yield a PD function that is convex in $F^{\prime}$.

## 6. Conclusions

The work in this paper was motivated by our desire to find a plausible answer to the question, what could the backorder penalty cost coefficient $b$ be? To this end, we proposed to infer the value of $b$ for the PB model by connecting $b$ to the loss in the long-run average demand rate which is affected by backorders according to Schwartz's PD model (1). We applied this procedure to the original PD
model and three variations of it in which we replaced the explicit fixed order cost with a constraint on the order quantity, the interorder time, and the starting inventory in each cycle, respectively. Our first main finding is that for the original PD model and the variation of the PD model with the minimum starting inventory in each cycle, the optimal fill rate is always 0 or 1 , which implies that the inferred backorder penalty cost $b$ in the respective PB models is 0 or $\infty$, respectively. In the former case, the optimal order quantity is infinite, whereas in the latter case it is finite. Based on the results in Section 5, our second main finding is that we have strong reasons to suspect that the property of $D^{\prime}\left(F^{\prime}\right)$ which is most likely responsible for producing this bang-bang type of result is strict convexity.

Future research following this work could be directed toward repeating this procedure for other PD models, for example models that assume that the long-run average demand rate is either a different function of the long-run average fill rate than the one given by equations (1) and (4), or a function of some other customer service related performance measure, such as the long-run average backorder waiting time or number of backorders.

Some such functions have been proposed in the literature. For example, Ernst and Cohen (1992) proposed a PD rate which is a linear function of the fill rate. Using our notation, their function can be written as

$$
D^{\prime}\left(F^{\prime}\right)=A\left[1-B\left(1-F^{\prime}\right)\right],
$$

where $A$ is the maximum potential demand rate corresponding to a fill rate equal to 1 and $B$ is a percentage.

Zipkin (2000) (problem 3.11, p. 69) proposed the PD function

$$
D^{\prime}\left(W^{\prime}\right)=\frac{a}{\left[p f\left(W^{\prime}\right)\right]^{\beta}},
$$

where $W^{\prime}$ is the average waiting time, $\alpha$ and $\beta$ are positive constants with $\beta>1$, and $f(\cdot)$ is an increasing function with $f(0)=1$. Given that the average waiting time can be expressed as a function of $Q^{\prime}$ and $F^{\prime}$ as well as the demand rate itself, namely

$$
W^{\prime}=\frac{Q^{\prime}\left(1-F^{\prime}\right)^{2}}{2 D^{\prime}\left(W^{\prime}\right)}
$$

if we substitute $W^{\prime}$ from the equation above into $D^{\prime}\left(W^{\prime}\right)$, we can see that the PD rate is a rather complicated function of $Q^{\prime}$ and $F^{\prime}$ satisfying

$$
D^{\prime}\left(Q^{\prime}, F^{\prime}\right)=\frac{a}{\left[p f\left(Q^{\prime}\left(1-F^{\prime}\right)^{2} / 2 D^{\prime}\left(Q^{\prime}, F^{\prime}\right)\right)\right]^{\beta}} .
$$

A less complicated alternative would be to replace the average waiting time $W^{\prime}$ with the average number of backorders, say $R^{\prime}$, in Zipkin's PD function, i.e., assume that

$$
D^{\prime}\left(R^{\prime}\right)=\frac{a}{\left[p f\left(R^{\prime}\right)\right]^{\beta}} .
$$

Given that the average number of backorders $R^{\prime}$ can be expressed as a function of $Q^{\prime}$ and $F^{\prime}$ as follows,

$$
R^{\prime}=\frac{Q^{\prime}\left(1-F^{\prime}\right)^{2}}{2},
$$

then $D^{\prime}(R)$ can be rewritten as a function of $Q^{\prime}$ and $F^{\prime}$ as follows:

$$
D^{\prime}\left(Q^{\prime}, F^{\prime}\right)=\frac{a}{\left[p f\left(Q^{\prime}\left(1-F^{\prime}\right)^{2} / 2\right)\right]^{\beta}} .
$$

In all the models above, the parameters of the PD function have to be estimated. As was mentioned in Section 4, Schwartz (1966) proposed a procedure for measuring parameters $A$ and $B$ in his model from observed demand data. In general, this is not a trivial task; however, it is a better defined task that picking a value for $b$. Finally, two other worthwhile directions for future research following this work would be to include direct backorder costs besides the indirect loss-of-customer-goodwill costs, to examine models with lost sales instead of order backlogging, and to extend this analysis to stochastic inventory models.

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## Appendix

Proof of Proposition 2: To solve the optimality conditions of the four optimization (maximization) problems corresponding to the four cases of the PD model shown in Table 3 of Section 4, we use Descartes's rule of signs, which was first published by Renée Descartes in 1637. This rule states that if the terms of a polynomial $f(x)$ are written in a customary fashion - that is with the terms given in decreasing order of the exponent of $x$ - then the number of positive real roots of the polynomial is either equal to the number of sign changes in the coefficients of successive terms of $f$
$(x)$ or is less than that number by an even number (until 1 or 0 is reached). If any coefficients are 0 , they are simply ignored. Similarly, the number of negative real roots of the polynomial is either equal to the number of sign changes in the coefficients of successive terms of $f(-x)$ or is less than that number by an even number (until 1 or 0 is reached) (e.g., see Young and Gregory, 1973).

Solution of the original PD model (case 1 of Table 3): In order to find the optimal order quantity as a function of $F^{\prime}, Q^{\prime *}\left(F^{\prime}\right)$, we set the first partial derivative of $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ with respect to $Q^{\prime}$ equal to 0 and solve the resulting equation. This equation is quadratic in $Q^{\prime}$ and has two solutions, one of which is negative. The only positive and therefore acceptable solution is

$$
\begin{equation*}
Q^{\prime *}\left(F^{\prime}\right)=\sqrt{\frac{2 k D^{\prime}\left(F^{\prime}\right)}{h F^{\prime 2}}}=\sqrt{\frac{2 k A}{h F^{\prime 2}\left[1+\left(1-F^{\prime}\right) B\right]}} . \tag{5}
\end{equation*}
$$

Let $P^{\prime^{*}}\left(F^{\prime}\right)$ be the average profit as a function of $F^{\prime}$ when the optimal order quantity is used, i.e.,

$$
\begin{equation*}
P^{\prime^{*}}\left(F^{\prime}\right)=P^{\prime}\left(Q^{\prime *}\left(F^{\prime}\right), F^{\prime}\right)=p D^{\prime}\left(F^{\prime}\right)-F \sqrt{2 k h D^{\prime}\left(F^{\prime}\right)}=\frac{p A}{1+\left(1-F^{\prime}\right) B}-F^{\prime} \sqrt{\frac{2 k h A}{1+\left(1-F^{\prime}\right) B}} . \tag{6}
\end{equation*}
$$

To find the optimal fill rate, $F^{\prime *}$, we set the first derivative of $P^{\prime *}\left(F^{\prime}\right)$ equal to 0 , solve the resulting equation, and examine the values of the average profit and its derivative at the end points of the interval $[0,1]$.

The first derivative of the average profit $P^{\prime^{*}}\left(F^{\prime}\right)$ is

$$
\begin{equation*}
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=\frac{p A B}{\left[1+\left(1-F^{\prime}\right) B\right]^{2}}-\frac{\sqrt{2 k h A}\left[2+\left(2-F^{\prime}\right) B\right]}{2 \sqrt{1+\left(1-F^{\prime}\right) B}} . \tag{7}
\end{equation*}
$$

Setting the first derivative of $P^{,^{*}}\left(F^{\prime}\right)$ equal to 0 , performing a change of variables from $F^{\prime}$ to $Y$, where $Y=1+\left(1-F^{\prime}\right) B$, and rearranging terms, yields the following $5^{\text {th }}$ degree polynomial equation in $Y$ :

$$
\begin{equation*}
Y^{5}+2(1+B) Y^{4}+(1+B)^{2} Y^{3}-\frac{2 A p^{2} B^{2}}{h k}=0 \tag{8}
\end{equation*}
$$

According to Descartes's rule of signs, the polynomial on the lhs of equation (8) has exactly one positive real root and exactly two or zero negative real roots. For each real root, $Y_{n}$, there corresponds a real root, $F_{n}^{\prime}$, of the rhs of expression (7), which is given by $F_{n}^{\prime}=1-\left(Y_{n}-1\right) / B$. Since $F^{\prime}$ represents the long-run, average fill rate, it must take values in the interval [ 0,1$]$. Note that if $Y_{n}<1$, then $F_{n}^{\prime}>1$, whereas if $Y_{n}>1+B$, then $F_{n}^{\prime}<0$. This implies that for each negative real root, $Y_{n}$, if there are any, the corresponding root $F_{n}^{\prime}$ is $>1$. It also implies that the root $F_{n}^{\prime}$ corresponding to the only positive real root, $Y_{n}$, lies in the interval [ 0,1 ] if and only if $Y_{n} \in[1,1+B]$. This means that at most one real root of the rhs of equation (7) may lie in the interval $[0,1]$.

With the above result in mind, to find the optimal fill rate, $F^{*^{*}}$, we proceed by examining the average profit and its derivative at the end points, 0 and 1 . From (7), it is easy to see that the first derivative of the average profit at the two end points, 0 and 1 , is given respectively by

$$
\begin{gather*}
\left.\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}\right|_{F^{\prime}=0}=\frac{p A B-(1+B) \sqrt{1+B} \sqrt{2 k h A}}{(1+B)^{2}},  \tag{9}\\
\left.\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}\right|_{F^{\prime}=1}=p A B-\sqrt{2 k h A}\left(1+\frac{B}{2}\right) \tag{10}
\end{gather*}
$$

From (6) it is also easy to see that the average profit at 0 and 1 is given respectively by

$$
\begin{gather*}
P^{\prime *}(0)=\frac{p A}{1+B},  \tag{11}\\
P^{\prime^{*}}(1)=p A-\sqrt{2 k h A} . \tag{12}
\end{gather*}
$$

Now, suppose that $P^{\prime^{*}}(0)>P^{,^{*}}(1)$, which from (11) and (12) is true if and only if $p A B<(B+1) \sqrt{2 h k A}$. The latter condition, which can be rewritten as $p \sqrt{A} B /(1+B)<\sqrt{2 h k}$, implies that the first derivative of the average profit at $F^{\prime}=0$, which is given by (9), is always negative. This means that as $F^{\prime}$ increases starting from 0 , the average profit, which starts at $P^{\prime^{*}}(0)$, either continuously decreases in the interval [ 0,1 ], or continuously decreases until it reaches a minimum at the only real root of the rhs of expression (7) which may possibly lie in the interval [0, 1], and then continuously increases - since there is at most one real root in the interval [0, 1] - until it reaches $P^{\prime^{*}}(1)$ at $F^{\prime}=1$. Given our initial assumption that $P^{\prime^{*}}(0)>P^{\prime^{*}}(1)$, this further implies that the maximum average profit in the interval $[0,1]$ is attained at $F^{\prime}=0$.

Now, suppose that $P^{\prime^{*}}(0)<P^{\prime^{*}}(1)$, which from (11) and (12) is true if and only if $p \sqrt{A} B /(1+B)>\sqrt{2 h k}$. Then, the first derivative of the average profit at $F^{\prime}=1$, which is given by (10), is always positive. This means that as $F^{\prime}$ decreases starting from 1 , the average profit, which starts at $P^{\prime^{*}}(1)$, either continuously decreases in the interval [ 0,1$]$, or continuously decreases until it reaches a minimum at the only real root of expression (7) which may possibly lie in the interval [0, $1]$, and then continuously increases - since there is at most one real root in the interval [ 0,1 ] - until it reaches $P^{\prime^{*}}(0)$ at $F^{\prime}=0$. Given our initial assumption that $P^{\prime^{*}}(0)<P^{\prime^{*}}(1)$, this further implies that the maximum average profit in the interval $[0,1]$ is attained at $F^{\prime}=1$.

Following the same argument, it can also be shown that if we assume that $P^{\prime^{*}}(0)=P^{\prime^{*}}(1)$, which from (11) and (12) is true if and only if $p \sqrt{A} B /(1+B)=\sqrt{2 h k}$, then the maximum average profit in the interval $[0,1]$ is attained at both $F^{\prime}=0$ and $F^{\prime}=1$.

## Solution of the PD model with a minimum order quantity (case 2 of Table 3)

For the PD model with a minimum order quantity $Q_{\min }$, in order to find the optimal order quantity $Q^{\prime *}$ note that the average profit function $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ is decreasing in $Q^{\prime}$; therefore, the optimal order quantity, $Q^{\prime *}$, should be as small as possible as long as the minimum order quantity constraint is not violated. This means that $Q^{* *}=Q_{\text {min }}$.

Let $P^{\prime^{*}}\left(F^{\prime}\right)$ be the average profit as a function of $F^{\prime}$ when the optimal order quantity is used, i.e.,

$$
\begin{equation*}
P^{\prime *}\left(F^{\prime}\right)=P^{\prime}\left(Q^{\prime *}, F^{\prime}\right)=P^{\prime}\left(Q_{\min }, F^{\prime}\right)=p D^{\prime}\left(F^{\prime}\right)-h \frac{Q_{\min } F^{\prime 2}}{2}=\frac{p A}{1+\left(1-F^{\prime}\right) B}-h \frac{Q_{\min } F^{\prime 2}}{2} \tag{13}
\end{equation*}
$$

To find the optimal fill rate, $F^{\prime^{*}}$, we set the first derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$ equal to zero, solve the resulting equation, and examine the values of the average profit and its derivative at the end points of the interval $[0,1]$.

The first derivative of the average profit $P^{,^{*}}\left(F^{\prime}\right)$, given by (13), is given by,

$$
\begin{equation*}
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=\frac{p A B}{\left[1+\left(1-F^{\prime}\right) B\right]^{2}}-h Q_{\min } F^{\prime} . \tag{14}
\end{equation*}
$$

The above expression implies that the first derivative of the average profit at $F^{\prime}=0$ is always positive. Setting the above expression equal to zero and rearranging terms yields the following cubic equation in $F^{\prime}$ :

$$
\begin{equation*}
h B^{2} Q_{\min } F^{\prime 3}-\left(2 h B Q_{\min }\right)(1+B) F^{\prime 2}+h Q_{\min }(1+B)^{2} F^{\prime}-p B A=0 . \tag{15}
\end{equation*}
$$

According to Descartes's rule of signs, the lhs of the above equation has exactly three or one real positive roots, and no negative real roots. To further investigate how many of the positive real roots lie in the interval $[0,1]$, we perform a change of variables from $F^{\prime}$ to $Y$, where $Y=1-F^{\prime}$, and reset expression (14) equal to zero. After rearranging terms we obtain the following cubic equation in $Y$ :

$$
\begin{equation*}
h B^{2} Q_{\min } Y^{3}+h B Q_{\min }(2-B) Y^{2}+2 h Q_{\min }(0.5-B) Y+p B A-h Q=0 . \tag{16}
\end{equation*}
$$

For each real root, $Y_{n}$, of the cubic polynomial on the lhs of the above equation, there corresponds a real root of the lhs of equation (15), $F_{n}^{\prime}$, which is given by $F_{n}^{\prime}=1-Y_{n}$. To determine how many of the roots $F_{n}^{\prime}$ lie in $[0,1]$, we proceed as follows.

First, suppose that $p B A<h Q_{\min }$. Then, according to Descartes's rule of signs, it can be easily shown that the lhs of the equation (16) has exactly one positive real root and two or zero negative real roots, regardless of the value of $B$. This is done by examining the cases where $B$ is less than 0.5 , $B$ is between 0.5 and 2 , and $B$ is greater than 2. This implies that for each negative real root $Y_{n}$, if any, the corresponding root $F_{n}^{\prime}$ is greater than one. It also implies that the root $F_{n}^{\prime}$ corresponding to the only positive real root $Y_{n}$ is less than one. Given than the lhs of equation (15) has no negative real roots, as was mentioned above, this further implies that the root $F_{n}^{\prime}$ corresponding to the only
positive real root $Y_{n}$ is also greater than zero. To summarize, if $p B A<h Q_{\text {min }}$, the cubic polynomial on the lhs of equation (15) always has exactly on real root, say $F_{R}^{\prime}$, in the interval $[0,1]$ and two or zero real roots which are greater than one. Moreover, given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, as was mentioned above, then as $F^{\prime}$ increases starting from zero, the average profit continuously increases in the interval $\left[0, F_{R}^{\prime}\right.$ ), reaches a maximum at $F^{\prime} R$, and decreases in the interval $\left(F_{R}^{\prime}, 1\right]$, since there are no other real roots in the interval $[0,1]$. This also means that the first derivative of the average profit at $F^{\prime}=1$ is negative, which from (14) is true if and only if $p A B<h Q_{\min }$. The latter condition coincides with our original assumption.

Next, suppose that $p B A>h Q_{\min }$ and $B<0.5$. Then, according to Descartes's rule of signs, the lhs of the equation (16) has no positive real roots and exactly three or one negative real roots. This implies that for each negative real root $Y_{n}$ the corresponding real root $F_{n}^{\prime}$ is greater than one. In other words, there are no real roots $F_{n}^{\prime}$ that lie in the interval $[0,1]$. Given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, as was mentioned above, this further implies that as $F^{\prime}$ increases starting from zero, the average profit continuously increases in the interval [0, 1] since there are no real roots in the interval $[0,1]$ - reaching a maximum at $F^{\prime}=1$. This also means that the first derivative of the average profit at $F^{\prime}=1$ is positive, which from (14) is true if and only if $p A B>h Q_{\text {min }}$. The latter condition coincides with our original assumption.

Finally, suppose that $p B A>h Q_{\min }$ and $B>0.5$. Then, according to Descartes's rule of signs, the lhs of the equation (16) has exactly two or zero positive real roots and one negative real root. This implies that the real root $F_{n}^{\prime}$ corresponding to the only negative real root $Y_{n}$ is greater than one. It also implies that for each positive real root $Y_{n}$, if any, the corresponding real root $F_{n}^{\prime}$ is less than one. Given than the lhs of equation (15) has no negative real roots, as was mentioned above, this further implies that the roots $F_{n}^{\prime}$ corresponding to the positive real roots $Y_{n}$, if any, are also greater than zero. To summarize, if $p B A>h Q_{\min }$ and $B>0.5$, the cubic polynomial on the lhs of equation (15) always has two or zero real roots in the interval [ 0,1$]$ and one real root which is greater than one. If there are two real roots in the interval $[0,1]$, then, given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, it is straightforward to see that the smallest root, say $F^{\prime}$, yields a local maximum of the average profit and the second root yields a local minimum. In this case, if $P^{\prime^{*}}\left(F_{2}^{\prime}\right)>P^{\prime^{*}}(1)$, which from (13), can be rewritten as $P^{\prime^{*}}\left(F_{2}^{\prime}\right)>p A-h Q_{\min } / 2$, then the average profit is maximized at $F^{\prime}=F^{\prime} 2$ in the interval $[0,1]$; otherwise, it is maximized at $F^{\prime}=1$. On the other hand, if there are no real roots in the interval $[0,1]$, then, given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, it is straightforward to see that the average profit is maximized at $F^{\prime}=1$ in the interval $[0,1]$.

The above analysis was extended to include the cases where $p B A=h Q_{\min }$ and $/$ or $B=0.5$. We omit the details here for space considerations.

## Solution of the PD model with a minimum interorder time (case 3 of Table 3)

For the PD model with a minimum interorder time $T_{\min }$, in order to find the optimal order quantity $Q^{\prime \prime *}$ note that the average profit function $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ is decreasing in $Q^{\prime}$, so the optimal order quantity, $Q^{\prime *}$, should be as small as possible as long as the minimum order quantity constraint is not violated. This means that $Q^{\prime *}=D\left(F^{\prime}\right) T_{\text {min }}$.

Let $P^{,^{*}}\left(F^{\prime}\right)$ be the average profit as a function of $F^{\prime}$ when the optimal order quantity is used, i.e.,

$$
\begin{align*}
P^{\prime *}\left(F^{\prime}\right) & =P^{\prime}\left(Q^{\prime *}, F^{\prime}\right)=P^{\prime}\left(D^{\prime}\left(F^{\prime}\right) T_{\min }, F^{\prime}\right)=p D^{\prime}\left(F^{\prime}\right)-h \frac{D^{\prime}\left(F^{\prime}\right) T_{\min } F^{\prime 2}}{2} \\
& =\frac{p A}{1+\left(1-F^{\prime}\right) B}-\frac{h A T_{\min } F^{\prime 2}}{2\left[1+\left(1-F^{\prime}\right) B\right]} . \tag{17}
\end{align*}
$$

To find the optimal fill rate, $F^{*^{*}}$, we set the first derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$ equal to zero, solve the resulting equation, and examine the values of the average profit and its derivative at the end points of the interval $[0,1]$.

The first derivative of the average profit $P^{\prime^{*}}\left(F^{\prime}\right)$, given by (17), is

$$
\begin{equation*}
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=\frac{p A B}{\left[1+\left(1-F^{\prime}\right) B\right]^{2}}-\frac{h A T_{\min } F^{\prime}}{1+\left(1-F^{\prime}\right) B}-\frac{h A B T_{\min } F^{\prime 2}}{2\left[1+\left(1-F^{\prime}\right) B\right]^{2}} . \tag{18}
\end{equation*}
$$

The above expression implies that the first derivative of the average profit at $F^{\prime}=0$ is always positive. Setting the above expression equal to zero and rearranging terms yields the following quadratic equation in $F^{\prime}$ :

$$
\begin{equation*}
h A B T_{\min } F^{\prime 2}-\left(2 h A T_{\min }\right)(1+B) F^{\prime}+2 p A B=0 . \tag{19}
\end{equation*}
$$

The above equation has the following two solutions:

$$
\begin{equation*}
1+\frac{1}{B} \pm \sqrt{\frac{(1+B)^{2}}{B^{2}}-\frac{2 p}{h^{2} T_{\min }^{2}}} \tag{20}
\end{equation*}
$$

If the term under the square root in the above expression is negative, then both solutions are complex numbers. In order for the term under the square root to be negative, the following condition must hold:

$$
2 p B^{2}>h(1+B)^{2} T_{\min }
$$

Suppose that the above condition does hold. Then equation (19) has no real solutions. Given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, as was mentioned above, then as $F^{\prime}$ increases starting from zero, the average profit continuously increases since equation (19) has no
real solutions; therefore, the maximum average profit for values of $F^{\prime}$ in the interval $[0,1]$ is attained at $F^{\prime}=1$.

If the term under the square root in expression (20) is positive, then both solutions of the quadratic equation (19) are real numbers; however, one of them is always greater than one. The only real solution that may lie in the interval $[0,1]$ is the solution $F^{\prime}$ given by (3). Given that the first derivative of the average profit at $F^{\prime}=0$ is always positive, then in order for the above solution to lie in the interval $[0,1]$, the first derivative of the average profit at $F^{\prime}=1$ must be negative. From (18), the latter is true if and only if $2 p B<h(2+B) T_{\text {min }}$, which can be rewritten as $2 p B /(2+B)<$ $h T_{\text {min. }}$. If this condition holds, then the maximum average profit for values of $F^{\prime}$ in the interval $[0,1]$ is attained at $F^{\prime}=F^{\prime}$; otherwise it is attained at $F^{\prime}=1$.

## Solution of the PD model with a minimum starting inventory (case 4 of Table 3)

For the PD model with a minimum starting inventory $I_{\text {min }}$, in order to find the optimal order quantity $Q^{\prime *}$ note that the average profit function $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ is decreasing in $Q^{\prime}$, so the optimal order quantity, $Q^{\prime^{*}}$, should be as small as possible as long as the minimum order quantity constraint is not violated. This means that $Q^{\prime *}\left(F^{\prime}\right)=I_{\text {min }} / F^{\prime}$.

Let $P^{\prime^{*}}\left(F^{\prime}\right)$ be the average profit as a function of $F^{\prime}$ when the optimal order quantity is used, i.e.,

$$
\begin{equation*}
P^{\prime^{* *}}\left(F^{\prime}\right)=P^{\prime}\left(Q^{\prime *}, F^{\prime}\right)=P^{\prime}\left(I_{\min } / F^{\prime}, F^{\prime}\right)=p D^{\prime}\left(F^{\prime}\right)-h \frac{I_{\min } F^{\prime}}{2}=\frac{p A}{1+\left(1-F^{\prime}\right) B}-h \frac{I_{\min } F^{\prime}}{2} . \tag{21}
\end{equation*}
$$

To find the optimal fill rate, $F^{\prime^{*}}$, we examine the first and second derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$, as well as the values of $P^{\prime^{*}}\left(F^{\prime}\right)$ at the end points of the interval $[0,1]$.

The first and second derivative of the average profit $P^{\prime *}\left(F^{\prime}\right)$, given by (21), are

$$
\begin{gather*}
\frac{d P^{\prime^{*}}\left(F^{\prime}\right)}{d F^{\prime}}=\frac{p A B}{\left[1+\left(1-F^{\prime}\right) B\right]^{2}}-\frac{h I_{\min }}{2},  \tag{22}\\
\frac{d P^{\prime * 2}\left(F^{\prime}\right)}{d F^{\prime 2}}=\frac{2 p A B^{2}}{\left[1+\left(1-F^{\prime}\right) B\right]^{3}} . \tag{23}
\end{gather*}
$$

From equation (23), it is obvious that for every $F^{\prime} \in[0,1]$, the second derivative of $P^{r^{*}}\left(F^{\prime}\right)$ is always positive. This means that the average profit is convex in $F^{\prime}$ in the interval $[0,1]$; therefore, the optimal fill rate, $F^{\prime^{*}}$, coincides with one of the two end points, 0 or 1 . More specifically, if $P^{\prime^{*}}$ $(0)>P^{\prime^{*}}(1)$, which from (21) is true if and only if $2 p A B /(1+B)<h I_{\min }$, then $F^{,^{*}}=0$. Conversely, if $P^{\prime^{*}}(0)<P^{\prime^{*}}(1)$, which from (21) is true if and only if $2 p A B /(1+B)>h I_{\min }$, then $F^{\prime^{*}}=1$. Finally, if $P^{\prime^{*}}(0)=P^{r^{*}}(1)$, which from (21) is true if and only if $2 p A B /(1+B)=h I_{\min }$, both 0 and 1 are optimal.

Inferring bin the PB model from the PD model: To infer $b$ in the PB model, we set the optimal decision variables, $Q^{*}$ and $F^{*}$, in that model equal to the respective variables, $Q^{* *}$ and $F^{\prime^{*}}$, in the PD model, and solve for $b$. In order for the optimal fill rate $F^{*}$ in the PB model, which given in column 2 of Table 2, to be equal to the optimal fill rate $F^{* *}$ in the PD model, which is given in column 2 of Table $4, b$ must satisfy

$$
\begin{align*}
b & =h \frac{F^{*^{*}}}{1-F^{*^{*}}}, \text { in cases 1-3, }  \tag{24}\\
b & =h \frac{F^{\prime^{* 2}}}{1-F^{\prime^{* 2}}}, \text { in case } 4 . \tag{25}
\end{align*}
$$

The above expressions give the inferred value of $b$ in the PB model. These expressions imply that if $F^{*}=0$, then the inferred value of $b$ in the respective cases of the PB model is 0 . They also imply that if $F^{\prime^{*}}=1$, then the inferred value of $b$ in the respective cases of the PB model is infinite. Finally, if $F^{\prime *}$ is anywhere between 0 and 1 , then the inferred value of $b$ in the respective case of the PB model is finite. The exact inferred value of $b$ for all the cases of the PB model is shown in the second to last column of Table 4.

Proof of Proposition 3: To find the optimal order quantity as a function of $F^{\prime}, Q^{*^{*}}\left(F^{\prime}\right)$, we set the first partial derivative of $P^{\prime}\left(Q^{\prime}, F^{\prime}\right)$ with respect to $Q^{\prime}$ equal to 0 and solve the resulting equation. This equation is quadratic in $Q^{\prime}$ and has two solutions, one of which is negative. The only positive and therefore acceptable solution is

$$
\begin{equation*}
Q^{\prime *}\left(F^{\prime}\right)=\sqrt{\frac{2 k D^{\prime}\left(F^{\prime}\right)}{h F^{\prime 2}}} \tag{26}
\end{equation*}
$$

Let $P^{\prime^{*}}\left(F^{\prime}\right)$ be the average profit as a function of $F^{\prime}$ when the optimal order quantity is used, i.e.,

$$
\begin{equation*}
P^{\prime *}\left(F^{\prime}\right)=P^{\prime}\left(Q^{\prime *}\left(F^{\prime}\right), F^{\prime}\right)=p D^{\prime}\left(F^{\prime}\right)-F^{\prime} \sqrt{2 k h D^{\prime}\left(F^{\prime}\right)} . \tag{27}
\end{equation*}
$$

To find the optimal fill rate, $F^{\prime^{*}}$, we look at the first and second derivatives of $P^{*^{*}}\left(F^{\prime}\right)$, which are given by the following expressions:

$$
\begin{gather*}
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=\left(p-\frac{F^{\prime} \sqrt{2 k h}}{2 \sqrt{D^{\prime}\left(F^{\prime}\right)}}\right) \frac{d D^{\prime}\left(F^{\prime}\right)}{d F^{\prime}}-\sqrt{2 k h D^{\prime}\left(F^{\prime}\right)},  \tag{28}\\
\frac{d^{2} P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime 2}}=\left(p-\frac{F^{\prime} \sqrt{2 k h}}{2 \sqrt{D^{\prime}\left(F^{\prime}\right)}}\right) \frac{d^{2} D^{\prime}\left(F^{\prime}\right)}{d F^{\prime 2}}+\sqrt{\frac{2 k h}{D^{\prime}\left(F^{\prime}\right)}}\left(\frac{F^{\prime} \frac{d D^{\prime}\left(F^{\prime}\right)}{d F^{\prime}}}{4 D^{\prime}\left(F^{\prime}\right)}-1\right) \frac{d D^{\prime}\left(F^{\prime}\right)}{d F^{\prime}} . \tag{29}
\end{gather*}
$$

If $2 p \sqrt{D^{\prime}\left(F^{\prime}\right)} \leq F^{\prime} \sqrt{2 k h}$, for all $F^{\prime} \in[0,1]$, then from (28), $d P^{\prime^{*}}\left(F^{\prime}\right) / d F^{\prime}<0$, and therefore the maximum average profit in the interval $[0,1]$ is attained at $F^{\prime}=0$.

If $2 p \sqrt{D^{\prime}\left(F^{\prime}\right)}>F^{\prime} \sqrt{2 k h}$, for some $F^{\prime} \in[0,1]$, and $\quad d^{2} D^{\prime}\left(F^{\prime}\right) / d F^{\prime 2}>0 \quad$ and $F^{\prime}\left(d D^{\prime}\left(F^{\prime}\right) / d F^{\prime}\right) \geq 4 D^{\prime}\left(F^{\prime}\right)$, for all $F^{\prime} \in[0,1]$, then from (29), $d^{2} P^{\prime *}\left(F^{\prime}\right) / d F^{\prime 2}>0$, for all $F^{\prime} \in[0,1]$. This implies that the maximum average profit in the interval $[0,1]$ is attained at $F^{\prime}=0$ or 1 .

Proof of Proposition 4: First, consider the case where $n=1$. In this case, $D^{\prime}\left(F^{\prime}\right)=A F^{\prime}$. After replacing $D^{\prime}\left(F^{\prime}\right)$ by $A F^{\prime}$ in expressions (26)-(29), $Q^{\prime^{*}}\left(F^{\prime}\right), P^{\prime^{*}}\left(F^{\prime}\right)$, and the first and second derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$ are given by:

$$
\begin{gather*}
Q^{\prime *}\left(F^{\prime}\right)=\sqrt{\frac{2 k A}{h F^{\prime}}},  \tag{30}\\
P^{\prime^{*}}\left(F^{\prime}\right)=p A F^{\prime}-\sqrt{2 k h A F^{\prime 3}},  \tag{31}\\
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=A p-\frac{3 \sqrt{2 k h A F^{\prime}}}{2},  \tag{32}\\
\frac{d^{2} P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime 2}}=-\frac{3}{4} \sqrt{\frac{2 k h A}{F^{\prime}}} . \tag{33}
\end{gather*}
$$

From (32), the first derivative of $P^{,^{*}}\left(F^{\prime}\right)$ has one positive real root, namely,

$$
\begin{equation*}
F_{1}^{\prime}=2 A p^{2} / 9 k h . \tag{34}
\end{equation*}
$$

From (33), the second derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$ is negative for all $F^{\prime} \in[0,1]$. This means that $F_{1}^{\prime}$ is a point of local maximum. Since $F_{1}^{\prime}$ is the only positive root, it is also a point of global maximum.

If $2 A p^{2}<9 k h$, then $F_{1}^{\prime}<1$ and the maximum average profit in the interval [ 0,1$]$ is attained at $F_{1}^{\prime}$; therefore $F^{\prime *}=F_{1}^{\prime}$ and hence $Q^{,^{*}} \equiv Q^{\prime *}\left(F^{\prime^{*}}\right)=Q^{,^{*}}\left(F_{1}^{\prime}\right)$, which by (30) and (34), is equal to $3 \mathrm{k} / \mathrm{p}$.

If $2 A p^{2}=9 k h$, then $F_{1}^{\prime}=1$ and the maximum average profit in the interval $[0,1]$ is attained exactly at 1 ; therefore $F^{\prime^{*}}=1$ and hence $Q^{\prime *} \equiv Q^{\prime *}\left(F^{\prime^{*}}\right)=Q^{\prime *}(1)$, which by (30) is equal to $\sqrt{2 k A / h}$.

Finally, if $2 A p^{2}>9 k h$, then $F_{1}^{\prime}>1$, and $d P^{\prime *}\left(F^{\prime}\right) / d F^{\prime}>0$, for all $F^{\prime} \in[0,1]$. Hence the maximum average profit in the interval $[0,1]$ is attained at 1 ; therefore $F^{\prime *}=1$ and again $Q^{\prime *}\left(F^{\prime *}\right)=\sqrt{2 k A / h}$.

Next, consider the case where $n>1$. In this case, $D^{\prime}\left(F^{\prime}\right)=A F^{m}$. After replacing $D^{\prime}\left(F^{\prime}\right)$ by $A F^{m}$, in expressions (26)-(29), $Q^{,^{*}}\left(F^{\prime}\right), P^{\prime^{*}}\left(F^{\prime}\right)$, and the first and second derivative of $P^{*^{*}}\left(F^{\prime}\right)$ are given by:

$$
\begin{gather*}
Q^{\prime *}\left(F^{\prime}\right)=\sqrt{\frac{2 k A F^{\prime n-2}}{h}},  \tag{35}\\
P^{\prime *}\left(F^{\prime}\right)=F^{\prime(n+2) / 2}\left(p A F^{\prime(n-2) / 2}-\sqrt{2 k h A}\right),  \tag{36}\\
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=F^{\prime n / 2}\left(n p A F^{\prime(n-2) / 2}-\left(\frac{n+2}{2}\right) \sqrt{2 k h A}\right),  \tag{37}\\
\frac{d^{2} P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime 2}}=F^{\prime(n-2) / 2}\left(n(n-1) p A F^{\prime(n-2) / 2}-\frac{n}{2}\left(\frac{n+2}{2}\right) \sqrt{2 k h A}\right) . \tag{38}
\end{gather*}
$$

From (37), the first derivative of $P^{\prime^{*}}\left(F^{\prime}\right)$ has two nonnegative real roots, namely, 0 and

$$
\begin{equation*}
F_{n}^{\prime}=\left(\frac{(n+2) \sqrt{2 k h A}}{2 n p A}\right)^{2 /(n-2)} . \tag{39}
\end{equation*}
$$

From (38),

$$
\begin{gather*}
d^{2} P^{*^{2} 2}\left(F^{\prime}\right) /\left.d F^{\prime 2}\right|_{F^{\prime}=0}=0,  \tag{40}\\
d^{2} P^{\prime * 2}\left(F^{\prime}\right) /\left.d F^{\prime 2}\right|_{F^{\prime}=F_{n}^{\prime}}=\frac{n(n-2) p A}{2} F_{n}^{(n-2)} . \tag{41}
\end{gather*}
$$

If $n>2$, then from (41), $d^{2} P^{\prime * 2}\left(F^{\prime}\right) /\left.d F^{\prime 2}\right|_{F^{\prime}=F_{n}^{\prime}}>0$. This implies that as $F^{\prime}$ decreases, starting from $F_{n}^{\prime}, P^{\prime *}\left(F^{\prime}\right)$ increases until $F^{\prime}$ reaches 0 ; similarly, as $F^{\prime}$ increases, starting from $F_{n}^{\prime}, P^{\prime *}\left(F^{\prime}\right)$ increases. This further implies that $F^{\prime^{*}}$ is either 0 or 1 , depending on whether $P^{\prime^{*}}(0)>P^{\prime^{*}}(1)$ or not. More specifically, if $P^{\prime^{*}}(0)>P^{,^{*}}(1)$, which from (36) implies that $A p^{2}<2 k h$, then $F^{\prime^{*}}=0$. In this case, $Q^{\prime *} \equiv Q^{\prime^{*}}\left(F^{\prime *}\right)=Q^{\prime *}(0)=0$, which from (35) is equal to 0 . If $P^{\prime^{*}}(0)<P^{\prime^{*}}(1)$, which from (36) implies that $A p^{2}>2 k h$, then $F^{\prime^{*}}=1$. In this case, $Q^{\prime *} \equiv Q^{\prime *}\left(F^{\prime *}\right)=Q^{\prime *}(1)$, which from (35) is equal to $\sqrt{2 k A / h}$. Finally, if $P^{\prime^{*}}(0)=P^{\prime^{*}}(1)$, which implies that $A p^{2}=2 k h$, then $F^{\prime^{*}}$ is either 0 or 1 , in which case, $Q^{\prime *}$ is 0 or $\sqrt{2 k A / h}$, respectively.

Finally, if $n=2$, then from (37),

$$
\begin{equation*}
\frac{d P^{\prime *}\left(F^{\prime}\right)}{d F^{\prime}}=2 F^{\prime}(p A-\sqrt{2 k h A}) . \tag{42}
\end{equation*}
$$

From (42), if $A p^{2}<2 k h$, then $d P^{\prime^{*}}\left(F^{\prime}\right) / d F^{\prime}<d P^{\prime^{*}}\left(F^{\prime}\right) /\left.d F^{\prime}\right|_{F^{\prime}=0}$ for all $F^{\prime} \in(0,1]$; therefore, $F^{\prime^{*}}=0$. On the other hand, if $A p^{2}>2 k h$, then $d P^{\prime *}\left(F^{\prime}\right) / d F^{\prime}>d P^{\prime^{*}}\left(F^{\prime}\right) /\left.d F^{\prime}\right|_{F^{\prime}=0}$ for all $F^{\prime} \in(0,1] ;$
therefore $F^{\prime^{*}}=1$. Finally, if $A p^{2}=2 k h$, then $d P^{\prime^{*}}\left(F^{\prime}\right) / d F^{\prime}=0$ for all $F^{\prime} \in[0,1]$; therefore, any fill rate in $[0,1]$ is optimal, i.e., $F^{\prime^{*}}=[0,1]$. Also, $Q^{,^{*}} \equiv Q^{\prime^{*}}\left(F^{\prime *}\right)$, which from (35) is equal to $\sqrt{2 k A / h}$, irrespectively of $F^{\prime *}$.

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