Buckling and post-buckling of long pressurized elastic thin-walled tubes under in-plane bending

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Abstract

The present paper focuses on the structural stability of long uniformly pressurized thin elastic tubular shells subjected to in-plane bending. Using a special-purpose non-linear finite element technique, bifurcation on the pre-buckling ovalization equilibrium path is detected, and the post-buckling path is traced. Furthermore, the influence of pressure (internal and/or external) as well as the effects of radius-to-thickness ratio, initial curvature and initial ovality on the bifurcation moment, curvature and the corresponding wavelength, are examined. The local character of buckling in the circumferential direction is also demonstrated, especially for thin-walled tubes. This observation motivates the development of a simplified analytical formulation for tube bifurcation, which considers the presence of pressure, initial curvature and ovality, and results in closed-form expressions of very good accuracy, for tubes with relatively small initial curvature. Finally, aspects of tube bifurcation are illustrated using a simple mechanical model, which considers the ovalized pre-buckling state and the effects of pressure.

Keywords: Tube; Cylindrical shell; Thin-walled; Ovalization; Stability; Bifurcation; Buckling

1. Introduction

The present work investigates the stability of long elastic thin-walled tubes subjected to in-plane bending and pressure (internal or external). Under pure bending loading conditions, the tube cross-section is distorted (ovalizes), resulting in loss of bending stiffness and a limit point instability, referred to as “ovalization” or Brazier instability. The bending moment at the limit point was calculated by Brazier [1] equal to

$$M_{BR} = 0.987Et^2r/\sqrt{1 - \mu^2},$$

where $r$ and $t$ are the cross-sectional radius and thickness, respectively, $E$ is Young’s modulus and $\mu$ is Poisson’s ratio. However, the increased axial stress at the compression side, accentuated because of ovalization causes bifurcation instability (buckling) in a form of longitudinal wavy-type “wrinkles”, usually before a limit moment is reached. This constitutes a buckling problem associated with a highly non-linear pre-buckling state, where the compression zone of the cylindrical shell wall has a double curvature in the longitudinal and in the hoop direction. In the recent years, this problem has received significant attention due to its application in nanomechanics; more specifically, several attempts have been reported to apply shell stability concepts in order to simulate the structural stiffness and explain the buckling and post-buckling response of carbon nanotubes subjected to bending loads [2–4].

In an early publication, Seide and Weingarten [5] investigated bifurcation of initially straight circular tubes under bending, assuming a linear (non-deformed) pre-buckling state, and a Ritz-type bifurcation solution in terms of trigonometric functions. Their numerical results indicated that the critical (buckling) moment $M_{cr}$ of a cylinder corresponds to a nominal bending stress $\sigma_{cr} = M_{cr}/(\pi r^2 t)$, which is quite close to the buckling stress of the cylinder under uniform compression, thus the following equations may offer a good approximation for the buckling moment and the corresponding bending stress:

$$M_{cr} = 1.813 \frac{Et^2r}{\sqrt{(1 - \mu^2)}},$$

(1)

$$\sigma_{cr} = \frac{E}{\sqrt{3(1 - \mu^2)}} \left( \frac{t}{r} \right).$$

(2)
Kempner and Chen [6], considering the DMV shell equations and assuming a linear pre-buckling state, examined bifurcation instability of circular and oval cylinders under bending, in the presence of axial force. Furthermore, they employed Koiter–Budiansky initial post-buckling theory [7,8] to investigate post-buckling behavior, in terms of trigonometric functions in the longitudinal direction, and obtained an asymptotic approximation of the secondary path, which indicated a symmetric bifurcation point, and an initially unstable secondary equilibrium path.

The solutions in [5] and [6] neglect cross-sectional ovalization on the pre-buckling state and, therefore, they predict an unreasonably high value of buckling moment for long tubes. The non-linear effects of the ovalized pre-buckling configuration on the bending response of tubes were considered by Axelrad [9]. In this work, it was assumed that bifurcation occurs when the maximum compressive bending stress value reaches the critical stress value for a uniformly compressed circular tube of radius equal to the local radius of the ovalized shell at the “critical” point. By consequence, Eq. (2) may be applicable in a local sense, replacing the initial hoop curvature \(1/r\) with the current curvature in the hoop direction at the location where buckling initiates. Using the same concept, Emmerling [10] computed the bifurcation bending moment and curvature of initially oval cylinders under bending and pressure.

Stephens et al. [11], using a finite-difference discretization of shell stability equations, investigated bifurcation of finite-length, initially-straight tubes under bending, considering pre-buckling ovalization, as well as end-effects, and calculated bifurcation moments for different levels of pressure. The buckling moment of long tubes was calculated close to Brazier ovalization limit moment \(M_{BR} = 0.987\pi^2EI/\sqrt{1-\mu^2}\). Fabian [12] examined buckling of thin elastic initially straight tubes under bending and pressure (internal or external), through a perturbation of the non-linear DMV shallow-shell equations. Results from the linearized (first-order) stability problem indicated that bifurcation occurs on the primary path before the ovalization limit point, regardless the level of pressure. Subsequently, using Koiter–Budiansky initial post-buckling theory, as described in [13] for non-linear pre-buckling state, and assuming a trigonometric variation of stress and displacements in the longitudinal direction, an asymptotic approximation of the post-buckling path was obtained.

Ju and Kyriakides [14] reported few numerical results for bending buckling of initially straight non-pressurized elastic tubes \((r/t = 100)\) using the non-linear Sanders’ shell equations and discretization in terms of trigonometric functions. The results verified that bifurcation occurs before the limit moment is reached and it is attained on the primary path. In addition, an unstable post-buckling path was observed. In a recent paper [15], using a special-purpose finite element formulation, instabilities of non-pressurized elastic tubes with radius-to-thickness ratio equal to 120 were investigated, with emphasis on the effects of relatively small initial curvature. Furthermore, it was concluded that, depending on the initial curvature value and the direction of bending load, buckling may occur before or after the limit point of the primary ovalization path. It was also found that depending on the hoop curvature and longitudinal stress variation, buckling may occur at various locations around the cross-section.

The present paper extends the investigation presented in [15], and examines the stability of initially-ovalized thin elastic tubes subjected to combined pressure and bending loading. The tubes are infinitely long and may be initially straight or bent with a longitudinal radius of curvature \(R\) significantly larger than the cross-sectional radius \(r\) \((R/r > 80)\). The dependence of buckling moment, curvature, wavelength and post-buckling response on the level of pressure, as well as on the value of initial curvature and initial ovality is examined in detail. Furthermore, the influence of tube thickness on the response is examined, considering the cross-sectional radius-to-thickness ratio \((r/t)\) to range between 20 and 720. Buckling configurations are calculated and discussed, and post-buckling paths are traced with particular emphasis on the location of buckling patterns around tube circumference. Motivated by the small size of the buckling zone in the hoop direction, a simplified analytical bifurcation solution is also developed, which employs a variational ovalization solution, presented in the Appendix, and results in closed-form expressions for the buckling curvature, the buckling moment and the corresponding wavelength. The analytical expressions are compared with numerical results. Finally, using a simple mechanical model, proposed elsewhere, enhanced to account for the pre-buckling ovalized state and the effects of pressure, the bifurcation response of tubes is illustrated in a qualitative manner.

2. Numerical technique

Numerical results are obtained through a non-linear variational formulation of the elastic shell, outlined in this section, based on a special-purpose “tube” finite element. The formulation, was introduced in [16] for the analysis of inelastic thick-walled cylinders, and has also been successfully employed in [15] for thin-walled elastic tubes.

2.1. Formulation and finite element discretization

A Lagrangian formulation of the tube is adopted using convected coordinates in the hoop, longitudinal and radial direction (denoted as \(\theta, \zeta, \rho\), respectively), as described in detail by Needleman [17]. The tube is considered as an isotropic hyper-elastic continuum. The constitutive equations relate the convected rate of Kirchhoff stress to the rate-of-deformation tensor. Following classical shell theory [8], the traction component normal to any shell lamina is imposed to be zero and tube thickness is assumed constant. The present formulation accounts for pressure loading, which is applied on the tube surface as a follower distributed load, always normal to the deformed shape of the tubular shell wall.

Discretization of the elastic continuum is considered through the three-node “tube element”, which combines longitudinal (beam-type) with cross-sectional deformation (Fig. 1). Nodes are located on the tube axis, which lies on the plane of bending,
and each node possesses three degrees of freedom (two translational and one rotational). A reference line is chosen within the cross-section at node \((k)\) and a local Cartesian coordinate system is defined, so that the \(x, y, z\) axes define the cross-sectional plane. The orientation of node \((k)\) is defined by the position of three orthonormal vectors \(e^x_k\), \(e^y_k\) and \(e^z_k\). For in-plane (ovalization) deformation, fibers initially normal to the reference line remain normal to the reference line. Furthermore, those fibers may rotate in the out-of-plane direction by angle \(\gamma(\theta)\). Using quadratic interpolation in the longitudinal direction, the position vector \(x(\theta, \zeta, \rho)\) of an arbitrary point at the deformed configuration is

\[
x(\theta, \zeta, \rho) = \sum_{k=1}^{3} (x^{(k)} + r^{(k)}(\theta) + \rho n^{(k)}(\theta)) + \rho \gamma(\theta)e^z_k N^{(k)}(\zeta),
\]

(3)

where \(x^{(k)}\) is the position vector of node \((k)\), \(r^{(k)}(\theta)\) is the position of the reference line at a certain cross-section relative to the corresponding node \((k)\), \(n^{(k)}(\theta)\) is the “in-plane” outward normal of the reference line at the deformed configuration and \(N^{(k)}(\zeta)\) is the corresponding Lagrangian quadratic polynomial. Using non-linear ring theory [8], vector functions \(r^{(k)}(\theta)\) and \(n^{(k)}(\theta)\), can be expressed in terms of the radial, tangential and out-of-plane displacements of the reference line, denoted as \(w(\theta)\), \(v(\theta)\), \(u(\theta)\) and \(\gamma(\theta)\) are discretized as follows:

\[
w(\theta) = a_0 + a_1 \sin \theta + \sum_{n=2,4,6,\ldots} a_n \cos n\theta
+ \sum_{n=3,5,7,\ldots} a_n \sin n\theta,
\]

(4)

\[
v(\theta) = -a_1 \cos \theta + \sum_{n=2,4,6,\ldots} b_n \sin n\theta
+ \sum_{n=3,5,7,\ldots} b_n \cos n\theta,
\]

(5)

\[
u(\theta) = \sum_{n=2,4,6,\ldots} c_n \cos n\theta + \sum_{n=3,5,7,\ldots} c_n \sin n\theta,
\]

(6)

\[
\gamma(\theta) = \sum_{n=0,2,4,6,\ldots} \gamma_n \cos n\theta + \sum_{n=1,3,5,7,\ldots} \gamma_n \sin n\theta.
\]

(7)

Coefficients \(a_n, b_n\) refer to in-plane cross-sectional deformation, and express the ovalization of the cross-section, whereas \(c_n, \gamma_n\) refer to out-of-plane (“warping”) cross-sectional deformation.

### 2.2. Numerical implementation

A 16th degree expansion \([n \leq 16\) in Eqs. (4)–(7)] for \(w(\theta), v(\theta), u(\theta)\) and \(\gamma(\theta)\) is found to be adequate for the purposes of the present study. Regarding the number of integration points in the circumferential direction, the 16th degree expansion requires 23 equally spaced integration points around the half-circumference. Three and two Gauss points are used in the radial (through the thickness) direction and the longitudinal direction of the “tube” element, respectively.

A periodic solution along the tube is considered for the post-buckled configuration and, therefore, only the tube portion corresponding to a half wavelength \(L_{hw}\) is analyzed, with appropriate boundary conditions at the two ends that enforce the periodicity of the solution. The \(L_{hw}\) value is not known a priori and, therefore, a sequence of analyses is conducted for each case considered, so that the actual wavelength is determined, i.e. the one that corresponds to the “earliest” bifurcation point on the primary ovalization path. Regarding the number of elements in the longitudinal direction, four elements per half-wavelength are found to be adequate for the purposes of the present work.

The non-linear governing equations are solved through an incremental Newton–Raphson numerical procedure, enhanced to enable the tracing of post-buckling “snap-back” equilibrium paths through an arc-length algorithm, which monitors the value of the so-called “arc-length parameter” [18]. This parameter, denoted \(\Delta l\) is a combination of bending moment increment \(\Delta M\) with the increment of some “selected” degrees of freedom \(\Delta u\).
as follows

\[ \Delta I^2 = \frac{\Delta u^T \Delta u}{\Delta u_r^T \Delta u_r} + \beta \left( \frac{\Delta M}{\Delta M_e} \right)^2, \] (8)

where \( \Delta M_e \) is a normalization moment, and \( \Delta u_r \) are normalization displacements, both obtained from a preliminary load-controlled small step. In Eq. (8), the incremental values of the two translational degrees of freedom of the “tube” element nodes are employed to form the vector of incremental displacements \( \Delta u \) and reference displacements \( \Delta u_r \). Furthermore, the value of \( \beta \) is taken equal to 1 (“spherical arc-length”).

To enable the incremental analysis to follow the post-buckling path, a very small initial imperfection of the tube is imposed. The initial imperfection is considered in the form of the buckling mode, obtained by an eigenvalue analysis just prior to bifurcation, and it is very small yet sufficient to improve convergence near the buckling point and to “trigger” bifurcation.

3. Numerical results

Results for relatively-thin infinitely-long elastic pressurized tubes are obtained, which focus on bifurcation instability and post-buckling response. In all cases, pressure—if present—is applied first and then, keeping the pressure level constant, bending load is gradually increased. Negative and positive values of pressure refer to internal and external pressure, respectively. Bending moments are either “closing” (in the direction of the initial curvature) or “opening” (opposite to the direction of the initial curvature). The values of pressure \( p_c \), moment \( M \) and longitudinal stress \( \sigma \) are normalized by \( p_{cr}, M_c \) and \( \sigma_c \), respectively, where \( p_{cr} = E t^3 / 4 r^3 (1 - \mu^2), M_c = E r t^2 / \sqrt{1 - \mu^2} \) and \( \sigma_c = E t / (r \sqrt{1 - \mu^2}) \), so that \( f = p / p_{cr}, m = M / M_c \). Curvature is expressed as the ratio of the relative rotation between the two end sections of the model over their initial distance, which is equal to \( L_{hw} \). The applied curvature \( k \), the initial curvature \( k_{in} \), and the total curvature \( k_T \) are normalized by \( k_N = t / (r^2 \sqrt{1 - \mu^2}) \), so that \( \kappa = k / k_N, \kappa_{in} = k_{in} / k_N \) and \( \kappa_T = k_T / k_N \), respectively. Poisson ratio \( \mu \) is equal to 0.3 in all cases analyzed. Cross-sectional ovalization is expressed through the following dimensionless ovalization parameter:

\[ \zeta = \frac{D_2 - D_1}{2D}, \] (9)

where \( D_1 \) is the diameter normal to the plane of bending and \( D_2 \) is the diameter on the plane of bending. Furthermore, the value of half-wave length \( L_{hw} \) is normalized by \( L_0 = \sqrt{r t} [\pi^2 / 12 (1 - \mu^2)]^{1/4} \), so that \( s = L_{hw} / L_0 \), where \( L_0 \) is the half-wavelength of an axisymmetrically-deformed elastic cylinder subjected to uniform axial compression [8].

In Fig. 2, the bending response of two non-pressurized initial straight tubes with \( r / t = 20 \) and \( r / t = 720 \) is plotted, and bifurcation occurs before a limit point is reached on the primary path. The path denoted as “uniform ovalization” corresponds to a cross-sectional bending analysis, using only one tube element and restraining all warping degrees of freedom, i.e. \( c_n = \gamma_n = 0 \) in Eqs. (6) and (7). Fig. 3 shows the dependence of buckling point on the \( r / t \) ratio. Thin-walled tubes (i.e. tubes with large values of \( r / t \)) buckle at smaller values of curvature and moment (\( k_{cr} \) and \( m_{cr} \)). In all cases, the initial post-buckling behavior is unstable, characterized by a “snap-back” immediately after bifurcation, which is reminiscent of the initial post-buckling path of circular or oval cylinders under uniform axial compression. The “snap-back” of the post-bifurcation path is sharper for thinner tubes.

The pre-buckling (just prior to bifurcation) and post-buckling configurations of deformed cylinders for zero pressure (\( r / t = 120 \)) are depicted in Fig. 4. Note that for visualization purposes, the post-buckling displacements are magnified. Upon bifurcation, the compressed part of the tube surface exhibits a wavy pattern, and the amplitude of the buckle waves increases along the secondary path. Another important observation concerns the “local” character of buckling in the circumferential direction;
Fig. 3. Bifurcation of unpressurized initially straight tubes ($f = 0$, $\kappa_{in} = 0$); effect of $r/t$ ratio on the $m$-$k$ equilibrium path.

Fig. 4. Pre-buckling and post-buckling shape of tube cross-sections in the absence of pressure ($\kappa_{in} = 0$, $f = 0$, $r/t = 120$).

Fig. 5. Bifurcation instability of initially straight tubes ($r/t = 120$) with respect to the level of pressure.
Fig. 6. Ovalization of initially straight tubes ($r/t = 120$) with respect to the level of pressure.

Fig. 7. Variation of critical and ovalization moments with respect to the pressure level for initially straight tubes ($r/t = 120$).

Fig. 8. Variation of critical and ovalization curvatures with respect to the pressure level for initially straight tubes ($r/t = 120$).
Fig. 9. Dependence of buckling half-wavelength \( s = \frac{L_{hw}}{L_0} \) on pressure level; initially straight tubes \( r/t = 120 \).

Fig. 10. Normalized strain energy curve in terms of applied curvature for initially straight, non-pressurized tubes \( f = 0, \kappa_0 = 0, r/t = 720 \); point \( (*) \) on the curve denotes bifurcation.

Fig. 11. Response of initially bent tube for closing moments for three different levels of pressure \( r/t = 120 \).
Fig. 12. Ovalization of initially bent tube for closing moments for three different levels of pressure ($r/t = 120$); points ($♦$) denote bifurcation.

Fig. 13. Response of initially bent tube under opening moments for three different levels of pressure ($r/t = 120$); thick line corresponds to path with buckling and thin line to uniform ovalization path, respectively.

Fig. 14. Ovalization of initially bent tube for opening moments for three different levels of pressure ($r/t = 120$); points ($♦$) denote bifurcation.
buckling occurs within a zone around the critical point, referred to as “buckling zone”. In the present case ($\kappa_{in}=0$, $\zeta_0=0$, $f=0$), the buckling zone is located in the vicinity of $\theta = \pi/2$, and its size depends on the tube thickness. Numerical calculations show that it decreases with increasing values of $r/t$ ratio. In Fig. 4 the size of the buckling zone, defined as the distance between the two “nodal points” A and B is equal to $0.69r$ ($r/t = 120$). Note that for $r/t = 20$ and $r/t = 720$ the corresponding sizes are calculated equal to $1.22r$ and $0.52r$, respectively.

The shape of Fig. 4 also indicates that post-buckling configuration is associated with an “inward” post-buckling displacement of the “buckling zone” uniform along the tube, and this is in agreement with experimental observations from uniformly compressed circular and oval cylinders [19].

The response of initially straight tubes with radius-to-thickness ratio ($r/t$) equal to 120, for different pressure levels is shown in Figs. 5 and 6, where the thick lines corresponds to paths with buckling, whereas the thin lines represents uniform ovalization response. In all cases, bifurcation occurs before a limit point is reached on the primary path, whereas the initial post-buckling path is unstable, exhibiting a “snap-back” immediately after bifurcation. The presence of external pressure results in a significant increase of cross-sectional ovalization (flattening) and causes a significant reduction of the buckling moment ($m_{ct}$) and the corresponding critical curvature ($\kappa_{cr}$). On the other hand, internal pressure alleviates cross-sectional ovalization and increases both the $m_{ct}$ and $\kappa_{ct}$ values. For high levels of internal pressure (e.g. $f = -10$), ovalization is negligible, the pre-buckling $m-\kappa$ path is quasi-linear and the $m_{ct}$ value approaches the buckling moment computed from Eq. (1) under the assumption of undeformed tube cross-section ($m_{ct} = 1.813$). Fig. 7 shows the variation of ovalization limit moment ($m_{ov}$) and critical moment ($m_{ct}$) with respect to the pressure level ($f$). Furthermore, the dependence of the
corresponding normalized curvature values \((\kappa_{ov} \text{ and } \kappa_{cr})\) on the pressure level \(f\) is plotted in Fig. 8, and indicates that, with increasing external pressure, the bifurcation point approaches the ovalization limit point. The dependence of buckling half-wavelength on the level of pressure is plotted in Fig. 9. The value of \(s\) for large values of internal pressure \((f \rightarrow -\infty)\) approaches unity \((s \rightarrow 1)\), which means that for high internal pressure the buckling wavelength becomes equal to the axisymmetric-buckling wavelength of a similar elastic cylinder subjected to uniform axial compression. On the other hand, for external pressure values close to \(p_{cr}\) \((f \rightarrow 1)\) the half wavelength value approaches infinity \((s \rightarrow \infty)\). In this case, the ovalization mechanism, accentuated by the presence of high external pressure, governs tube response.

In Fig. 10, the elastic deformation energy of the tube per unit length is plotted in terms of curvature for an initially non-pressurized straight tube \((\kappa_{in} = 0, f = 0)\). The energy is normalized by the product of \(M_e\) and \(k_N\). The diagram is initially monotonically increasing and exhibits a negative “jump” at the bifurcation curvature. This discontinuity is more pronounced for a thin-walled tube \((r/t = 720)\), shown in the detail of Fig. 10. Beyond this point, it continues to increase monotonically. The reason for the discontinuity of the diagram is the “snap-back” of the initial post-bifurcation path immediately after buckling. Note that experimental measurements, as well as molecular dynamics simulations in elastic carbon nanotubes, have shown a similar “kink” on the elastic deformation energy diagram [4].

The bending response of circular initially slightly bent tubes \((\kappa_{in} = \pm 0.20)\) in terms of the pressure level \((f)\) is shown in Figs. 11–15. Negative and positive values of \(\kappa_{in}\) correspond to opening and closing bending moments, respectively. For this value of initial curvature, buckling occurs before the ovalization limit moment, regardless the level of pressure and the post-buckling path is also characterized by a “snap-back”. The numerical calculations also show that the “buckling zone” is located around \(\theta = \pi/2\) for both closing and opening moments.
The moment-curvature plots in Figs. 11 and 13 indicate that the presence of external pressure accentuates cross-sectional ovalization and therefore, reduces the moment capacity $m_{cr}$ and the corresponding critical curvature $\kappa_{cr}$. On the other hand, there is a beneficial effect of internal pressure on the $m_{cr}$ and $\kappa_{cr}$ values, due to the significant reduction of cross-sectional ovalization, as shown in Figs. 12 and 14. Note that, in the case of opening moments (Fig. 14), the tube initially exhibits reverse ovalization (negative values of $\zeta$), so that the diameter on the plane of bending lengthens and the other principal diameter shortens (“bulging” ovalization) until the total curvature of the tube becomes about half the initial curvature value ($\kappa_T \simeq -|\kappa_{in}|/2$). Subsequently, “bulging” ovalization decreases and, beyond the curvature where the tube becomes straight ($\kappa_T = \kappa + \kappa_{in} = 0$), cross-sectional “flattening” occurs until buckling. The fact that all $\kappa_T - \zeta$ curves pass from the

Fig. 19. Response of initially bent tube for opening moments ($\kappa_{in} = -1.374$) ($f = 0.5, 0.0, -0.5$, and $r/t = 120$).
Fig. 20. Ovalization of initially bent tube under opening moments ($\kappa_{in} = -1.374$) for three different levels of pressure ($r/t = 120$).

Fig. 21. Pre-buckling and post-buckling shapes of cross-sections for three different pressure levels: (a) and (b) $f = 0$ critical point $\theta \approx \pi/6$, (c) $f = 0.5$ critical point $\theta \approx \pi/6$, (d) $f = -0.5$ critical point $\theta \approx 40^\circ$; ($\kappa_{in} = -1.374, r/t = 120$).
The response of circular tubes with more pronounced initial curvature ($\kappa_{in} = 1.030$) under closing bending moments is shown in Figs. 15 and 16. The numerical results indicate that bifurcation occurs well beyond limit point instability, for all three pressure levels, so that ovalization instability governs tube response. In addition, the secondary path under external pressure ($f = 0.65$) follows closely the primary equilibrium path. A closer view of the secondary path for $f = 0.65$ around origin ($\zeta = 0$) can be verified from the simplified ovalization-curvature Eq. (41) of the Appendix.

Fig. 22. Response of initially ovalized tube in the absence of pressure ($f = 0, \kappa_{in} = 0, r/t = 120$).

Fig. 23. (a) Bifurcation analysis of initially “bulged” tubes under for five different pressure levels ($r/t = 120, \kappa_{in} = 0$); (b) detail of equilibrium path at bifurcation for $f = 0.217$; (c) detail of equilibrium path at bifurcation for $f = 0.65$; arrows (↓) denote bifurcation.
bifurcation is shown in the detail of Fig. 15. The ovalization response of these pressurized tubes \((K_{\text{in}} = 1.030)\), plotted in Fig. 16, indicates that bucking occurs at large values of cross-sectional flattening \((\zeta_K > 0.35)\). The “flattened” cross-sectional shapes of the buckled tube configurations (Fig. 17), show that the wavy pattern also occurs within a small portion of the tube circumference, verifying the “local” character of buckling also observed in Fig. 4. Nevertheless, the \(\theta = \pi/2\) location may not be critical in all cases. In the absence of pressure \((f = 0)\), the critical point is located at about \(\theta = \pi/3\) (Figs. 17a, b). In the presence of external pressure \((f = 0.65)\) the numerical results indicated that the buckling zone is located at the “extrados” of the cross-section \((\theta = -\pi/2)\), as shown in Fig. 17c. This is explained by the compressive longitudinal stresses at \(\theta = -\pi/2\), depicted in the detail of Fig. 18, in conjunction with the flat shape of the ovalized cross-section at this location.

The response of an initially curved tube subjected to opening bending moments \((K_{\text{in}} = -1.374)\) is shown in Figs. 19 for different pressure levels. In all three cases buckling occurs before a limit point is reached. Furthermore, buckling occurs before the tube becomes straight \((K_T < 0)\), as shown in Fig. 20, and the corresponding post-buckling tube response exhibits a very sharp “snap-back”. The buckled tube cross-sections within a half-wavelength are depicted in Fig. 21, and show that for the three pressure levels, the buckling zone is no longer in the vicinity of \(\theta = \pi/2\). Also note that the externally pressurized case corresponds to the most pronounced bulging ovalization.

Finally, the effects of initial cross-sectional ovality on the buckling moment are examined, considering a relatively small stress-free doubly-symmetric out-of-roundness of the tube cross-section, which is assumed constant along the tube

\[
w_0(\theta) = \alpha_0 \cos 2\theta, \tag{10}
\]

\[
v_0(\theta) = -\frac{\alpha_0}{2} \sin 2\theta. \tag{11}
\]

The above expressions correspond to a “first-order inextensional” ovalization deformation \((dw_0/d\theta) + w_0 = 0\). Using the ovalization definition of Eq. (9), the ovalization parameter has an initial value \(\zeta_0\) equal to \(\alpha_0/r\). The effects of such an imperfection on the bending response of an initially straight tube \((K_{\text{in}} = 0)\) are shown in Fig. 22 for zero pressure \((f = 0)\), and for relatively small initial ovality \(|\zeta_0| \leq 0.1\). Positive values of initial ovalization correspond to “initial flattening” of the tube’s cross-section, whereas negative values refer to “initial bulging”. The results of Fig. 23a demonstrate that the orientation of the initial out-of-roundness may be quite important, especially in the presence of external pressure \((f > 0)\). In particular, reverse initial ovality, combined with external pressure, results in a post-buckling path that follows closely the primary equilibrium path (Figs. 23b, c). In Fig. 23, all the \(m-K\) curves, regardless the pressure level, pass through a common point A \((m_A = 0.945\) and \(K_A = 0.306)\) located before the bifurcation point. Furthermore, at this value of curvature the corresponding cross-sectional ovalization is zero (point A in Fig. 24). The above values of \(m_A\) and \(K_A\) can be also verified by the simplified analytical ovalization solution presented in the Appendix; requiring \(\zeta + \zeta_0 = 0\) and \(K_{\text{in}} = 0\) in Eqs. (41) and (42), one readily obtains \(K = |\zeta_0|^{1/2}\) and \(m = \pi|\zeta_0|^{1/2}\), and that for \(\zeta_0 = -0.1\) those values are very close to \(m_A\) and \(K_A\).

4. Simplified analytical solution

Simplified bifurcation analyses for shells have been proposed in previous works based on the assumption that buckling is fully determined by the stress and deformation inside the zone of the initial buckle, and that stresses and curvatures inside that zone are constant \([9, 12, 15, 20, 21]\). Under this assumption, an expression similar to Eq. (2) can be obtained, which governs shell instability at each point around the circumference. The local character of buckling around the cross-section shown in

Fig. 24. Ovalization of initially “bulged” tubes under for five different pressure levels \((r/t = 120)\).
**4.1. Linearized shell equations**

Local coordinates $X$ and $Y$ are defined on the buckling zone area, denoting the distances from the center of the buckling zone, in the longitudinal and the hoop direction, respectively, (Fig. 25). Starting from the non-linear DMV equations

$$
\frac{E t^3}{12(1-\mu^2)} \nabla^4 W - \frac{1}{R_x} N_x - \frac{1}{R_y} N_y \\
+ 2N_{xy} \frac{\partial^2 W}{\partial X \partial Y} = p(X, Y), 
$$

(12)

$$
\frac{1}{E t} \nabla^4 F - \frac{1}{R_0} \left( \frac{\partial^2 W}{\partial Y^2} \right) - \frac{1}{R_y} \left( \frac{\partial^2 W}{\partial X^2} \right) - \frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial Y^2} \\
- \left( \frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial Y^2} \right)^2 = 0, 
$$

(13)

where $W$ and $F$ are the displacement and stress functions, respectively, $N_x$, $N_y$, $N_{xy}$ are the membrane stress resultants (equal to the second derivatives of the stress function $F$), $1/R_x$, $1/R_y$ are the local curvatures of the deformed tube, and following the linearization procedure described in [21], the linearized form of Eqs. (12) and (13) is obtained as follows:

$$
[(\partial_x^2 + \partial_y^2)^2 + \frac{1}{E t h^6}(N_x \partial_x^2 + N_y \partial_y^2 - 2N_{xy} \partial_x \partial_y) (\partial_x^2 + \partial_y^2)^2 \\
+ (k_x \partial_x^2 + k_y \partial_y^2 - 2k_r r_{00} \partial_x \partial_y) (\partial_x^2 + \partial_y^2)^2] \tilde{w} = 0, 
$$

(14)

where $\tilde{w}(x, y)$ is a small deviation of the radial displacement from the pre-buckling state, $x$ and $y$ are dimensionless local coordinates, so that

$$
x = X/c, \quad y = Y/c, \quad c = \sqrt{r_{00}/}\sqrt{12(1-\mu^2)}, 
$$

(15)

$$
\tilde{c}_x, \tilde{c}_y \text{ denote partial derivatives with respect to } x \text{ and } y, 
$$

(16)

$$
k_0 = \frac{t}{r_{00}\sqrt{12(1-\mu^2)}}, \quad k_x = \frac{r_{00}}{R_x}, \quad k_y = \frac{r_{00}}{R_y} 
$$

(17)

where $A$ is an arbitrary constant and $n$ is a dimensionless wave number.

It is noted that a more general form of Eq. (18), considering trigonometric variation in both directions $x$ and $y$, is reported in [21]. Minimization of $N_x$ in Eq. (18) with respect to $n$ gives $n = 1$, or equivalently,

$$
L_{bw} = \frac{\pi c}{n} = \frac{\pi}{[12(1-\mu^2)]^{1/4} r_{00} t^3}, 
$$

(19)

and Eq. (18) becomes

$$
\sigma_{ax} = \frac{E}{\sqrt{3(1-\mu^2)} t r_{00}^3}, 
$$

(20)

where $\sigma_{ax}$ is the longitudinal buckling stress within the buckling zone. Eq. (20) resembles Eq. (2), and implies that the tube buckles at the location where stress $\sigma_{ax}$ becomes equal to the buckling stress of a uniformly compressed circular cylinder, with radius equal to the current hoop radius $r_{00}$ at the critical location.

**4.2. Closed-form solution**

In the present study, Eq. (20) is further elaborated to obtain a closed-form expression for the bifurcation curvature. The analysis is limited to tubes with relatively small initial curvature and initial ovality, so that buckling occurs at $\theta = \pi/2$. The key step in the development of the closed-form solution is consideration of the simplified ovalization solution presented in the Appendix, to describe the pre-buckling state.

In particular, the local hoop curvature $1/r_{00}$ and the longitudinal stress $\sigma_{ax}$ at $\theta = \pi/2$ at the ovalized pre-buckling configuration can be
Fig. 26. Variation of critical moment with respect to the pressure level; comparison between numerical results for three different $r/t$ ratios and analytical solution.

Fig. 27. Variation of critical curvature with respect to the pressure level; comparison of numerical results for three different $r/t$ ratios with analytical solution.

Fig. 28. Variation of buckling wavelength with respect to the pressure level; comparison of numerical results for three different $r/t$ ratios with analytical solution.
approximated from Eqs. (43), (44) as follows:

$$\frac{1}{r_{00}} = \frac{1}{r_0(\pi/2)} = \frac{-3\zeta_0 + 1 - f - 3\kappa(\kappa + \kappa_{in})}{r(1 - f)},$$

$$\sigma_{x0} = \sigma_x(\pi/2) = \frac{E}{\sqrt{1 - \mu^2}} \left( \frac{t}{t} \right) \left[ -\varepsilon_{x0} \frac{\kappa + f\kappa_{in}}{1 - f} + \kappa \left( 1 - \frac{(\kappa + \kappa_{in})^2}{1 - f} \right) \right].$$

Substituting the above expressions for $1/r_{00}$ and $\sigma_{x0}$ in the buckling Eq. (20), a third-order algebraic equation is obtained in terms of the critical curvature $\kappa_{cr}$:

$$\kappa_{cr} \left( 1 - \frac{(\kappa_{cr} + \kappa_{in})^2}{1 - f} \right) - \frac{1}{\sqrt{3}} \left( 1 - 3\kappa_{cr}(\kappa_{cr} + \kappa_{in}) \right)$$

$$\kappa_{cr} + f\kappa_{in} \frac{\kappa_{cr} + f\kappa_{in}}{1 - f} + \sqrt{3} \frac{\zeta_0}{1 - f} = 0$$

which has the following closed-form solution

$$\kappa_{cr} = \left[ \frac{1}{\sqrt{3}} - \frac{2\kappa_{in}}{3} + \frac{2}{3} \sqrt{abs[3(2 - f) - \sqrt{3}\kappa_{in} + \kappa_{in}^2 - 3\zeta_0]} \right]$$

$$\times \cos \left( \frac{\pi}{3} + \frac{1}{3} \arccos \left[ \frac{1}{2} \frac{3\sqrt{3}\kappa_{in}^2 - 2\kappa_{in}^3 + 6\sqrt{3}(1 - 6\zeta_0) + 9\kappa_{in}(3 - 2\zeta_0 + f(3 - 2\zeta_0))}{abs[3(2 - f) - \sqrt{3}\kappa_{in} + \kappa_{in}^2 - 3\zeta_0]^{3/2}} \right] \right).$$
where \(\text{abs}[\cdot]\) is the absolute value of \([\cdot]\). Subsequently, the bifurcation moment \(m_{cr}\) is obtained from Eq. (42) of the Appendix as follows:

\[
m_{cr} = -\frac{3}{4} \epsilon_0 r \left( \frac{2\kappa + f \kappa_{in}}{1 - f} \right) + \kappa_{cr} \left( 1 - \frac{3(\kappa_{cr} + \kappa_{in})(2\kappa_{cr} + \kappa_{in})}{4(1 - f)} \right).
\]

(25)

Expressions (24) and (25) define the bifurcation point on the primary \(m-\kappa\) path of equation. Furthermore, Eqs. (19) and (21) lead to the following closed-form expression for the normalized half wavelength:

\[
s = \sqrt{\frac{(1 - f)}{-3\epsilon_0 + (1 - f) - 3\kappa_{cr}(\kappa_{cr} + \kappa_{in})}}.
\]

(26)

For the particular case of initially straight pressurized tubes without initial ovality \((\kappa_{in} = 0 \text{ and } \epsilon_0 = 0)\), Eq. (23) becomes

\[
\kappa_{cr} \left( 1 - \frac{1 - f}{1 - f} \kappa_{cr}^2 \right) - \frac{1}{\sqrt{3}} \left( 1 - \frac{3}{1 - f} \kappa_{cr}^2 \right) = 0
\]

(27)

and its solution is written in closed form as follows:

\[
\kappa_{cr} = \frac{1}{\sqrt{3}} + \frac{2}{3} \sqrt{\text{abs}(3(2 - f))}
\times \cos \left( \frac{\pi}{3} + \frac{1}{3} \arccos \left[ \frac{3\sqrt{3}}{\text{abs}(3(2 - f))^{3/2}} \right] \right).
\]

(28)

Furthermore, the expression for the corresponding normalized half-wave length becomes:

\[
s = \frac{L_{bw}}{L_0} = \frac{r_0(1 - f)^{3/2}}{(1 - f) - 3\kappa_{cr}^2}.
\]

(29)

The accuracy of \(\kappa_{cr}\) and \(m_{cr}\) values, obtained from the above analytical equations, are shown in Figs. 26, 27, 28 for circular initially straight tubes, together with numerical results, indicating a remarkable accuracy. The agreement is better in the case of thin-walled tubes (large values of \(r/t\)), because those tubes exhibit a smaller size of buckling zone, as discussed in the previous section. The comparison between the numerical results with the analytical solution shows that the above closed-form expressions are quite accurate for relatively small values of \(\kappa_{in}\) and \(\epsilon_0\), as shown in Figs. 29 and 30. In those figures, Eq. (42) is used to express the pre-buckling analytical solution, and the bifurcation point, (denoted by the arrows (↓↑)) is obtained from Eqs. (24) and (25). For the particular case of zero initial ovality \((\epsilon_0 = 0)\) the expressions provide very good accuracy when \(-0.4 \leq \kappa_{in} \leq 0.2\).

5. Discussion

A simple model depicted in Fig. 31a, provides a physical explanation of the numerical and analytical results, reported in the previous sections. The model was proposed in an early publication [22] for cylindrical shells under uniform axial compression. According to this model, the tube is considered as a “bundle” of compressed longitudinal strips in the longitudinal direction, each one supported by a series of springs, so that the problem under consideration is similar to the buckling problem of a beam on elastic foundation. The “foundation springs” are elastic arches, representing the stiffness provided by the hoop deformation of the shell.

In the present study, the model is used to illustrate some bifurcation aspects of pressurized tubes under bending. More specifically, the compressed strip is considered in the middle of the buckling zone, and the arch may be ovalized, representing the shape of the tubular cross-section just prior to buckling.

![Diagram](https://example.com/diagram.png)

Fig. 31. (a) Simple mechanical model, simulating buckling of tubular shells under axial compression [22]. (b) Response of circular and initially oval arches under a concentrated load on the crest; results from shell finite element analysis [23]. Negative values of \(\delta_{in}\) indicate initial ovality opposite to the one shown in the sketches.
Arch stiffness plays a key role on the buckling stress and the corresponding wavelength, and depends on the amount of ovalization. In Fig. 31b, the response of elastic arches under a single concentrated load on the crest is plotted. Three cases are considered, corresponding to initially non-ovalized (circular) initially “flattened” and initially “bulged” tubes, denoted as cases A, B and C, respectively. The results are obtained numerically, using a non-linear degenerated shell finite element analysis [23]. In those results, the “flattening” direction of the load is considered positive. In all three cases, the response is non-linear “softening/hardening”, resulting in an unstable post-buckling path for the compressed strip, as discussed in [24]. Fig. 31b also shows that the support-arch stiffness is significantly reduced in the initially “flattened” arch (case B), but it is quite higher in the initially “bulged” arch (case C).

The above model can illustrate some aspects of tube bifurcation under the combined action of bending and pressure. In the case of closing bending moments, the cross-section flattens around the critical location, reduces the stiffness of the supporting arches, and results in a decrease of the critical moment, as shown in Figs. 11 and 15. The reduction of support stiffness is accentuated in the presence of external pressure, whereas internal pressure reduces flattening, increasing the stiffness of the supports and the corresponding critical longitudinal stress. Similarly, the presence of initial curvature opposite to the direction of bending (opening moments), increases the local hoop curvature, resulting in larger support stiffness, and therefore, it corresponds to a shorter wavelength and a higher critical moment, also depicted in Figs. 13 and 19.

6. Conclusions

In the present paper, the non-linear response of long pressurized thin-walled elastic tubes is examined, through a special-purpose, “tube” finite element formulation. The effects of pressure (internal and/or external), initial curvature and initial ovality, as well as the influence of the radius-to-thickness ratio are investigated. The results show that the response is governed by the strong interaction of cross-sectional ovalization, which characterizes the non-linear pre-buckling path, and bifurcation instability, in the form of uniform wrinkles along the tube. On the other hand, in the circumferential direction, buckling occurs within a limited region, called the “buckling zone”.

For circular, initially straight tubes, buckling occurs prior to ovalization instability, regardless the pressure level, the buckling zone is located at \( \theta = \pi/2 \), and bifurcation is followed by a “snap back” of the post-buckling (secondary) equilibrium path, which is more pronounced in the case of thin-walled tubes. In initially bent tubes under opening moments \( (k_{\text{in}} < 0) \) buckling occurs before ovalization limit point, for any pressure level, followed by a sharp “snap back”. On the other hand, for closing moments \( (k_{\text{in}} > 0) \) buckling may occur prior or beyond ovalization limit point depending on the value of initial curvature \( (k_{\text{in}}) \).

Generally, the presence of external pressure results in a reduction of the buckling moment and the corresponding buckling curvature, and increases the buckling wavelength value. On the other hand, the presence of internal pressure reduces cross-sectional ovalization, increases the buckling moment and decreases the corresponding wavelength. The effects of relatively small initial ovality are also examined in the presence of pressure. It is demonstrated that the orientation of initial ovality may have a significant influence. For the case of initial “flattening”, bifurcation occurs before a limit point is reached on the primary path, whereas for the case of initial “bulging”, bifurcation may occur prior or beyond the limit point depending on the pressure level.

A simplified analytical bifurcation solution is also presented, which results in closed-form expressions for the critical curvature, the critical moment and the corresponding buckling wavelength of thin-walled elastic tubes subjected to pressurized bending. The analytical formulation is based on the linearized DMV shell equations, employs the ovalization solution, presented in the Appendix, to describe the pre-buckling state, and uses the assumption that stress and deformation are constant within the buckling zone. The closed-form expressions provide results of remarkable accuracy with respect to the finite elements results, for relatively small values of initial curvature. Furthermore, the predictions of the closed-form expressions are closer to the numerical results for thin-walled tubes. Finally, the post-buckling behavior of elastic tubes is studied using a simple mechanical model, proposed elsewhere, and the effects of ovalization and pressure are illustrated in an elegant manner.

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Appendix. Closed-form ovalization solution

It is possible to derive a simplified closed-form ovalization solution that accounts for the effects of initial curvature, initial ovality and pressure, using a simple variational approach, which is described in this Appendix. The formulation considers the potential energy per unit length, expressed as follows:

\[
\Pi = U_L + U_C + V_P - W_P - Mk,
\]

where \( U_L \) is the longitudinal strain energy, \( U_C \) is the cross-sectional (hoop) strain energy due to ovalization, \( V_P \) is the pressure potential and \( W_P \) is the second-order work of hoop pressure stress \( (\sigma_P = pr/t) \). The longitudinal strain energy \( U_L \) is expressed in terms of longitudinal stress \( \sigma_x \) and strain \( \varepsilon_x \):

\[
U_L = \frac{1}{2} \int_A \sigma_x \varepsilon_x \, dA = \frac{E}{2} \int_A \varepsilon_x^2 \, dA = \frac{E tr}{2} \int_0^{2\pi} \varepsilon_x^2 \, d\theta,
\]

where \( A \) is the tube cross-section \( (A = 2\pi rt) \).

\[
\varepsilon_x = k\sqrt{\frac{\mu R}{E}} = k[(r + w_0 + w) \sin \theta + (w_0 + v) \cos \theta] + \frac{1}{R} [v \cos \theta + w \sin \theta].
\]
In the above expression, $y$ is the distance from the neutral axis, and $u_n$ is the displacement in the direction of the plane of bending. The initial ovality of the cross-section is described by stress-free displacements $w_0(\theta)$ and $v_0(\theta)$ in the radial and the tangential direction, respectively. The additional radial and tangential displacements are described by $w(\theta)$ and $v(\theta)$, respectively. The hoop strain is

$$\varepsilon_0 = \varepsilon_{0m} + k_\theta \rho = \left[ \frac{1}{r} (v' + w) + \frac{1}{r^2} (v' - w') \right] \rho,$$  

(33)

where $\varepsilon_{0m}$ is the membrane hoop strain, $k_\theta$ is the change of hoop curvature, and $(\cdot)'$ denotes differentiation with respect to $\theta$, so that

$$U_C = \frac{1}{2} \int_{-\ell/2}^{\ell/2} \int_0^{2\pi} \sigma_{\theta \theta} \varepsilon_{0m} \frac{dr}{d\rho} \, d\theta \, d\rho = \frac{1}{2} \frac{E r}{(1 - \rho^2)} \int_{-\ell/2}^{\ell/2} \int_0^{2\pi} \varepsilon_0^2 \, d\theta \, d\rho,$$  

(34)

It is readily shown [8] that the pressure potential $V_p$ is

$$V_p = p \Delta A = p(A^* - A_0),$$  

(35)

where $A^*$ is the area enclosed by the deformed ring

$$A^* = \pi r^2 + \frac{1}{2} \int_0^{2\pi} (2r w + v^2 + v' w - v w') \, d\theta,$$  

(36)

and $A_0$ is the area enclosed by the initially ovalized ring, equal to $\pi r^2$, and external pressure $p$ is assumed positive.

The second-order work of hoop pressure stress ($\sigma_p = pr/t$) is

$$W_p = \int_0^{2\pi} \frac{pr}{t} \, \varepsilon_{0m} \, r \, d\theta = \frac{1}{2} \frac{p r}{t} \int_0^{2\pi} ((v_0 + v) - (w_0 + w'))^2 \, d\theta,$$  

(37)

where $\varepsilon_{0m} = \frac{1}{2} ([v_0 + v] - [w_0 + w'])/r^2$ is the non-linear part of the membrane hoop strain. A simple “inextentional” Ritz-type solution for $w(\theta)$ and $v(\theta)$ is assumed

$$w(\theta) = \alpha \cos 2\theta \quad \text{and} \quad v(\theta) = -\frac{\alpha}{2} \sin 2\theta,$$  

(38)

which may be considered as a simplified version of expansions (4) and (5), with $a_2 = -2b_2$ and all other ovalization parameters equal to zero. The initial ovality of the tube’s cross-section is also assumed “inextentional” in the form of Eqs. (38)

$$w_0(\theta) = \alpha_0 \cos 2\theta \quad \text{and} \quad v_0(\theta) = -\frac{\alpha_0}{2} \sin 2\theta.$$  

(39)

In Eqs. (38) and (39) $\alpha_0$ and $\alpha$ are the initial and the additional ovalization amplitude, respectively.

Neglecting the quadratic terms of $\alpha$ in $U_L$, and considering Eqs. (38) and (39) one obtains

$$\Pi(\alpha, k) = \frac{E r t}{2} \left( k^2 r^2 - \frac{3}{2} k r (\alpha + \alpha_0) k + 2k_{m\text{in}} \right) + \frac{3}{8} \frac{E \pi r^2 \alpha^2}{(1 - \rho^2)^3} - \frac{3}{8} \frac{E \pi (\alpha + \alpha_0)^2}{(1 - \rho^2)^3} + \alpha (\alpha + 2\alpha_0)) - Mk,$$  

(40)

Minimization of $\Pi$ results in the following expressions:

$$\zeta = \frac{\alpha}{r} = \frac{\varepsilon_0 \alpha}{1 - f} + \frac{k (K + k_{m\text{in}})}{1 - f},$$  

(41)

where $\zeta$ is the normalized ovalization in the tube’s cross-section, additional to the initial one (so that $\zeta_T = \zeta_0 + \zeta$), and

$$m = -\zeta_0 r \left[ \frac{3(2K + K_{m\text{in}})}{4(1 - f)} \right] + \kappa \left[ \frac{1 - (3K + 3K_{m\text{in}})(2K + K_{m\text{in}})}{4(1 - f)} \right],$$  

(42)

that describes the primary path (uniform ovalization), considering the effects of pressure, initial curvature and initial ovality. Using Eqs. (39) and (41), closed-form expressions for the longitudinal stress ($\sigma_\alpha = E \varepsilon_\alpha$) and for the hoop curvature at the deformed configuration $(1/r_0)$ are readily obtained

$$\frac{\sigma_\alpha(\theta)}{\sigma_\alpha} = -\frac{\alpha_0}{(1 - f)} \left[ \frac{2K - 2K_{m\text{in}}}{4(1 - f)} \right] \frac{\sin \theta - \sin 3\theta}{\sin 3\theta},$$  

(43)

$$\frac{1}{r_0(\theta)} = \frac{1}{r} + \frac{3\alpha_0(1 - 2f)}{r(1 - f)} \cos 2\theta + \frac{3K + 3K_{m\text{in}}}{r(1 - f)} \cos 2\theta.$$  

(44)

Expressions (41)–(44) account for initial ovality, initial curvature and pressure. For zero initial ovality ($\zeta_0 = 0$) are similar to those derived in [25] for slightly bent tubes, using an asymptotic approximation of the non-linear ring equations. Furthermore, for the special case of zero initial ovality and zero pressure ($\zeta_0 = 0$, $f = 0$), a moment-curvature equation quite similar to Eq. (42) was derived in [26]. Expressions (41)–(44) compare quite well with numerical results up to the ovalization limit point of tubes, with relatively small initial curvature $(0.4 \leq K_{m\text{in}} \leq 0.2)$, as shown in Figs. 29 and 30, and can be used to describe the non-linear pre-buckling state in a simple and efficient manner in the case of thin-walled elastic tubes. More accurate semi-numerical solutions for ovalization bending of elastic tubes have been reported in [25–27] and, more recently, in [28].

References