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On the Tradeoff between Optimal Order-Base-Stock Levels and Demand Lead-Times

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We investigate the tradeoff between finished-goods inventory and advance demand information for a model of a single-stage make-to-stock supplier who uses an order-base-stock replenishment policy to meet customer orders that arrive a fixed demand lead-time in advance of their due-dates. We show that if the replenishment orders arrive in the order that they are placed, the tradeoff between the optimal order-base-stock level and the demand lead-time is "exhaustive," in the sense that the optimal order-base-stock level drops all the way to zero if the demand lead-time is sufficiently long. We then provide a sufficient condition under which this tradeoff is linear. We verify that this condition is satisfied for the case where the supply process is modeled as an M/M/1 queue. We also show that the tradeoff between the optimal order-base-stock level and the demand lead-time is linear for the case where the supply process is modeled as an M/D/1 queue. More specifically, for this case, we show that the optimal order-base-stock level decreases by one unit if the demand lead-time increases by an amount equal to the supplier's constant processing time. Finally, we show that the tradeoff between the optimal order-base-stock level and the demand lead-time is exhaustive but not linear in the case where the supply process is modeled as an M/D/∞ queue. We illustrate these results with a numerical example.

Keywords: *production/inventory system, make-to-stock queue, advance demand information, order-base-stock policy*

1 Introduction

There is a growing consensus among operations management researchers and practitioners that obtaining and distributing customer demand information to all the partners of a supply chain is essential for improving the coordination and ultimately the performance of the supply chain. The advantages of sharing customer demand information are further amplified when this information is obtained ahead of time. Such information is often referred to in the literature as *advance demand* (or *order*) *information* (ADI).

From a supplier's side, one of the main advantages of having access to ADI is that such information can be used as a tradeoff for finished-goods inventory and can thus lead to reduced inventory costs. The nature of this tradeoff, however, is in general difficult to assess, particularly when the supplier is a capacitated production/inventory system, because production capacity affects the tradeoff in a non-trivial way.

In this paper, we investigate the tradeoff between finished-goods inventory and ADI for a model of a single-stage make-to-stock supplier who uses a continuous-review order-base-stock replenishment policy to satisfy customer orders that arrive a fixed demand lead-time in advance of their due-dates. An order-base-stock policy works like a conventional base-stock or one-for-one replenishment policy except that replenishment orders are triggered by customer orders instead of actual demands. Its advantage is that it requires minimal information and is very simple to implement. Moreover, under some conditions it has been shown to be effective or even optimal, as is explained in Section 2.

A critical assumption in our model is that the supplier's replenishment orders arrive in the order that they are placed. We show that under this assumption, the tradeoff between the optimal order-base-stock level and the demand lead-time is "exhaustive," in the sense that the optimal order-base-stock level drops all the way to zero if the demand lead-time is sufficiently long. We then provide a sufficient condition under which this tradeoff is linear. We show that this condition is satisfied for the case where the supply process is modeled as an M/M/1 queue, thus verifying the results of Buzacott and Shanthikumar (1993, 1994), who first studied this case.

We also show that the tradeoff between the optimal order-base-stock level and the demand lead-time is linear for the case where the supply process is modeled as an M/D/1 queue. More specifically, for this system, we prove that if the demand lead-time increases by an amount equal to the supplier's constant processing time, then the optimal order-base-stock level decreases by one unit.

Finally, we derive a non-closed-form expression for the tradeoff between the optimal order-base-stock level and the demand lead-time for the case where the supply process is modeled as an M/D/∞ queue, which was first analyzed in detail by Hariharan and Zipkin (1995). We then show by a numerical example that for this system, the tradeoff may sometimes be concave (i.e. the higher the demand lead-time, the sharper the decrease in the optimal order-base-stock level), depending on the ratio of the inventory holding cost rate over the backorder cost rate.

The rest of this paper is organized as follows. In Section 2, we review some of the literature on ADI, particularly that which is most closely related to our work. In Section 3, we describe and analyze the general model of the supplier, and in Section 4, we apply and extend some of the results of this analysis to the special cases where the supplier is modeled as an M/D/1, M/M/1, and M/D/∞ queuing system, respectively. In Section 5, we illustrate these results with a numerical example, and in Section 6, we conclude.

2 Literature Review

The literature on ADI is in its early stages but is growing fast. One way of classifying it is based to whether the supply process is modeled as an uncapacitated inventory system or a capacitated production/inventory system.

Most of the literature on ADI concerns uncapacitated inventory systems. One of the earliest and most influential works for systems with exogenous replenishment times is the work of Hariharan and Zipkin (1995). They study a model of a supplier who uses a continuous-review order-base-stock replenishment policy to meet customer orders that arrive according to a Poisson process. Each customer order is for a single item to be delivered a fixed demand lead-time following the order. They consider three cases for modeling the demand and replenishment (i.e. supply) lead-times. For each case, they construct an equivalent conventional model, i.e. one with no demand lead-times, in which the replenishment lead-times are offset by the demand lead-times. This shows that the effect of a demand lead-time on overall system performance is the same as a corresponding reduction in the replenishment lead-time.

Gallego and Özer (2001) consider a single-stage periodic-review inventory system with exogenous replenishments and variable but finite demand lead-times. They show that for the zero set-up cost case, an order-base-stock policy is optimal if the replenishment time is greater than the maximum demand lead-time. Gallego and Özer (2003) and Özer (2003) extend this analysis to multi-echelon and distribution systems, respectively, and Wang and Toktay (2006) extend it to systems with flexible delivery. Finally, Özer and Wei (2004) prove the optimality of a state-dependent modified order-base-stock policy for an extension of the single-stage system in which the capacity is limited.

Other works that show the benefits of ADI on systems with exogenous replenishment times are Bourland et al. (1996), Güllü (1997), Decroix and Mookerjee (1997), Chen (2001), van Donselaar et al. (2001), Lu et al. (2003), Marklund (2006), and Tan et al. (2007).

There has been also a stream of related literature on forecast updating that was inspired by the dynamic forecast models of Graves et al. (1986) and Heath and Jackson (1994). Examples of such work are Güllü (1996), Toktay and Wein (2001), and Hu et al. (2003).

For queue-type capacitated production/inventory systems, Buzacott and Shanthikumar (1993, 1994) present a detailed model of a single-stage make-to-stock manufacturer who uses a continuous-review order-base-stock replenishment policy to meet customer demands that arrive a fixed demand lead-time in advance of their due-dates. They analyze in detail the case where demands arrive according to a Poisson process and the manufacturing system consists of a single server with exponentially distributed processing time and FCFS service protocol; hence, the flow through the manufacturing system is identical to that through an M/M/1 queue. For this system, they show that the optimal demand lead-time and associated cost is a linearly decreasing function of the order-base-stock level.

For the discrete-time version of the M/M/1 make-to-stock manufacturing system analyzed in Buzacott and Shanthikumar (1993, 1994), Karaesmen et al. (2002) evaluate analytically the performance of the optimal order-base-stock policy. They then compare it to the performance of the overall optimal replenishment policy, which they evaluate numerically using dynamic programming. Their numerical results show that the optimal order-base-stock policy is quite effective.

Karaesmen et al. (2003) complement the work of Buzacott and Shanthikumar (1993, 1994) with some results on the influence of production lead-time variability on the tradeoff between the order-base-stock level and the demand lead-time. Along the way, they propose an approximation scheme for a generalization of the model studied by Buzacott and Shanthikumar (1993, 1994) in which the flow through the manufacturing system is identical to that through an M/G/1 queue.

Karaesmen at al. (2004) assess the value of ADI for the model considered by Buzacott and Shanthikumar (1993, 1994) by assuming that the manufacturer pays a fixed or a demand lead-time-dependent price for obtaining ADI. They then evaluate the effects of processing capacity on the value of ADI. They repeat this assessment for a variation of the model in which customers accept deliveries earlier than their required due-dates. For this variation, they show that the effect of a demand lead-time on overall system performance is the same as a reduction in the backorder cost in an equivalent conventional system, i.e. one with no demand lead-times.

Liberopoulos et al. (2003) propose an order-base-stock-type policy for a model of a make-to-stock supplier with two classes of customers: those who provide unreliable ADI in the form of cancelable reservations, and those who provide no ADI at all. They optimize this policy via simulation.

Gayon et al. (2006) and Benjaafar et al. (2006) use Markov decision process analysis to characterize the structure of the optimal policy of a single-stage capacitated supply system with imperfect ADI, where customers either make cancelable reservations, as in the system introduced by Liberopoulos et al. (2003), or provide changeable due-dates, respectively.

Liberopoulos et al. (2005) investigate via simulation the tradeoff between the optimal order-base-stock levels and kanbans (WIP-control limits) and the demand lead-time, in order-base-stock policies with/without WIP-limits, for a single- and a two-stage make-to-stock capacitated manufacturing system with ADI.

Wijngaard (2004) considers a single-stage make-to-stock manufacturing system that either produces at a constant production rate R or not all. The goal is to meet customer orders with minimum average inventory and stockout costs; both cases of lost sales and order backlogging are considered. Customer orders arrive according to a Poisson process a fixed demand lead-time h in advance of their due-dates. The flow through the manufacturing system is therefore equivalent to that through an M/D/1 queue, except that production is continuous. The main result is that for high utilization rate ρ and small demand lead-times the finished-goods inventory reduction due to the foreknowledge of ADI is equal to $(1 - \rho)\ h\ R$.

Finally, Wijngaard and Karaesmen (2006) show that for the make-to-stock M/D/1-type queue considered in Wijngaard (2004), if the demand lead-time is smaller than a certain threshold value, then an order-base-stock policy is optimal. For unit production rate, this threshold value is equal to the optimal base-stock level for the case without ADI.

3 Model Description and Analysis

We consider a model of a make-to-stock supplier who sells a single type of items. Customer orders arrive for one item at a time according to a stationary stochastic process with mean arrival rate λ. Each customer order arrives a fixed demand lead-time, T, in advance of its requested due-date. Once a customer order arrives, it cannot be cancelled. If no items are available at the requested due-date, the customer's request is backordered. Customers do not accept early deliveries to avoid incurring inventory holding costs. There is no setup cost or setup time for placing a replenishment order and no limit on the number of orders that can be placed per unit time.

The supplier uses an order-base-stock replenishment policy to meet the demand. According to this policy, he places a replenishment order every time a customer order arrives. This way, he keeps his inventory position (number of in-process replenishment orders plus finished-goods inventory minus backorders) minus the in-process customer orders, i.e. the customers orders whose due-dates have not yet expired, equal to a specified fixed order-base-stock level, S.

The model described above is similar to that considered in Karaesmen at al. (2003, 2004). It is simple but it captures the basic operation of a make-to-stock supplier with constant reliable ADI, except for lot sizing and multiple-product issues, which we purposely keep out of the picture for simplicity, so as to focus our attention on ADI-related issues. A schematic representation of the model is shown in Figure 1, where the circles represent delays.

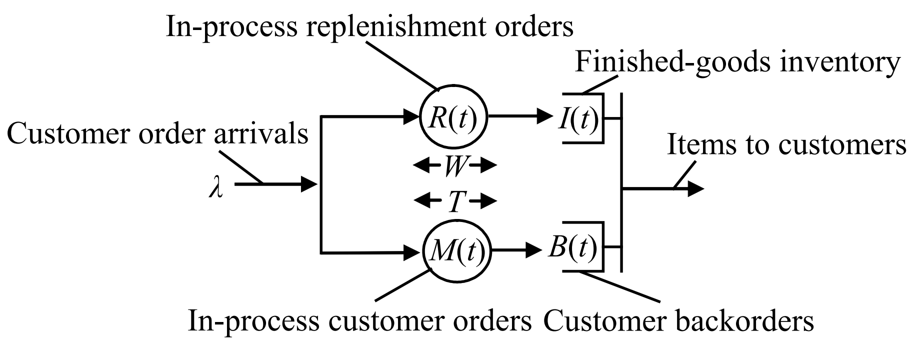

Figure 1: Model of a make-to-stock supplier operating under an order-base-stock policy to respond to customer orders with a fixed demand lead-time.

In Figure 1 and in the rest of the paper, we use the following notation:

$I(t)$ = number of items in finished-goods inventory at time t,

$B(t)$ = number of customer backorders at time t,

$R(t)$ = number of in-process replenishment orders at time t,

$M(t)$ = number of in-process customer orders at time t,

$X(t) = I(t) - B(t)$ = finished-goods surplus/backlog at time t,

$Z(t) = R(t) - M(t)$ = in-process replenishment orders surplus/backlog at time t,

By definition of the order-base-stock policy, it is easy to see that the above quantities satisfy the invariant

$$Z(t) + X(t) = R(t) - M(t) + I(t) - B(t) = S. \tag{1}$$

In what follows, we will use $P_Y(\cdot)$ and $F_Y(\cdot)$ to denote the stationary probability distribution function and cumulative distribution function of a random variable $Y(t)$, and $\overline{F}_Y(\cdot)$ to denote its complementary cumulative distribution function, i.e. $\overline{F}_Y(y) = 1 - F_Y(y)$.

Let h and b denote the unit costs per unit time of holding finished-goods inventory and customer backorders, respectively. We consider a standard optimization problem whose objective is to find the order-base-stock level S that minimizes the long-run expected average cost of holding finished-goods inventory and customer backorders, as a function of T. The long-run expected average cost function is identical to that of the newsvendor problem; therefore, for a given value of T, the optimal order-base-stock level, $S^*(T)$, can be found – as in the newsvendor problem – from the first order optimality condition. This condition implies that

$$S^*(T) = \arg\min_{S:S\,\text{int}} \left\{ F_Z(S) \geq b/(h+b) \right\}. \tag{2}$$

Note that if an integer S satisfies the condition in (2) with equality, i.e. if $F_Z(S) = b/(h + b)$, then both S and $S + 1$ are optimal order-base-stock levels when the demand lead-time is T.

If $T = 0$, there is no delay between the arrival of a customer order and the request for the delivery of an item from finished-goods inventory. In this case, $M(t) = 0$, and hence $Z(t) = R(t)$. Then, from (2), the optimal order-base-stock level, $S^*(0)$, which in this case is simply equal to the optimal base-stock level in the standard newsvendor model, i.e. without ADI, is given in terms of $F_R(\cdot)$ by the familiar expression

$$S^*(0) = \arg\min_{S:S\,\text{int}} \left\{ F_R(S) \geq b/(h+b) \right\}. \tag{3}$$

If $T > 0$, then $M(t) \neq 0$ and hence $Z(t) \neq R(t)$. In this case, in order to compute $S^*(T)$ from (2) we need to know $P_Z(\cdot)$. Proposition 1, which follows, gives an expression of $P_Z(\cdot)$ in terms of $P_R(\cdot)$.

Proposition 1. *The stationary probability distribution function of* $Z(t)$ *is given by*

$$P_Z(n) = \begin{cases} \sum_{k=n}^{\infty} P_{G_T^k}(k-n) P_R(k), & n > 0, \\ \sum_{k=0}^{\infty} P_{G_T^k}(k-n) P_R(k), & n \leq 0, \end{cases} \tag{4}$$

where $P_{G_T^k}(i)$ *is the stationary conditional probability of having i replenishment completions in a time interval of length T, given that at the beginning of this time interval there are k in-process replenishment orders in the system.*

The proof of Proposition 1 is based on the same arguments that Buzacott and Shantikumar (1993) use for the special case where the supplier is modeled as an M/M/1 queue.

Obtaining $P_{G_T^k}(\cdot)$ and consequently $P_Z(\cdot)$ is not trivial; therefore, computing $S^*(T)$ from (2) is not an easy task. Intuition suggests that as T increases starting from zero, $S^*(T)$ should decrease. The question that we address in this paper is how exactly does it decrease?

To try to answer this question, let us switch the space of our analysis from inventories to time. Consider the moment of a customer order arrival. As a convention, define this arrival as the initial or zeroth arrival and set its time equal to zero. Let H_1 denote the time that elapses between this, i.e. the initial, and the next, i.e. the first, customer order arrival. Moreover, for $n = 2, 3,\ldots$, let H_n denote the time that elapses between the $(n-1)$th and the nth arrival thereafter. Suppose that the times H_n, $n = 1, 2,\ldots$, are i.i.d. random variables with known cumulative distribution function, $F_H(\cdot)$, and mean, $E[H] = 1/\lambda$, where λ is the mean arrival rate.

For $n = 0, 1,\ldots$, let A_n denote the time of the nth customer order arrival. From the above definitions, it is easy to see that

$$A_0 = 0 \text{ and } A_n = A_{n-1} + H_n = \sum_{i=1}^{n} H_i, \quad n = 1, 2,\ldots, \tag{5}$$

From (5) it is also easy to see that $A_n > A_{n-1}$, $n = 1, 2,\ldots$

As was mentioned above, the use of an order-base-stock policy by the supplier means that as soon as a customer order arrives, the supplier places a replenishment order. Let W_n denote the replenishment time of the nth order. Suppose that the times W_n, $n = 0, 1,\ldots$, are identically distributed random variables with known cumulative distribution function $F_W(\cdot)$ and mean $E[W] = L$.

Let E_n denote the difference between the replenishment time triggered by the initial order, W_0, and the arrival time of the nth customer order, A_n, i.e.

$$E_n = W_0 - A_n, \quad n = 0, 1,\ldots \tag{6}$$

From (5) and (6), it is easy to see that

$$E_0 = W_0 \text{ and } E_n = W_0 - \sum_{i=1}^{n} H_i, \quad n = 1, 2,\ldots, \tag{7}$$

$$E_n = E_{n-1} - H_n, \quad n = 1, 2,\ldots \tag{8}$$

Expression (7) implies that the cumulative distribution function of E_n, $F_{E_n}(\cdot)$, depends on $F_W(\cdot)$ and, if $n > 0$, on $F_H(\cdot)$ as well. From (8), it is easy to see that E_n is stochastically larger than E_{n-1}, i.e.

$$F_{E_n}(T) > F_{E_{n-1}}(T), \quad n = 1, 2,\ldots \text{ and } T \geq 0. \tag{9}$$

Moreover, by definition of a cumulative distribution function,

$$F_{E_n}(T + \Delta T) > F_{E_n}(T), \quad n = 0, 1,\ldots, T \geq 0, \text{ and } \Delta T > 0. \tag{10}$$

In what follows, we will suppose that the following assumption holds.

Assumption 1. *All replenishment orders enter the supply system one at the time, remain in the system until they are fulfilled (there is no blocking, balking or reneging), leave one at a time in the order of arrival (FIFO) and do not affect the flow time of previous replenishment orders (lack of anticipation).*

Assumption 1 implies that the replenishment order that is triggered by the nth customer order arrival at time A_n will satisfy the $(n + S)^{\text{th}}$ customer order. Assumption 1 leads to the following proposition.

Proposition 2. *Under Assumption 1, the optimal order-base-stock level for a given demand lead-time T, $S^*(T)$, is given by*

$$S^*(T) = \arg\min_{S:S\,\text{int}} \left\{ F_{E_{S+1}}(T) \geq b/(h+b) \right\}. \tag{11}$$

The proof of Proposition 2 is based on the same arguments that Karaesmen at al. (2004) use to prove their Proposition 1. That proposition, which is slightly more restrictive than Proposition 2, states that when the supplier operates in a make-to-order mode, i.e. with a zero order-base-stock level, then the optimal demand lead-time is given by

$$T^*\big|_{S=0} = \arg\min_{T} \left\{ F_W(T) \geq b/(h+b) \right\}. \tag{12}$$

The above expression is in fact identical to expression (14) of Corollary 2 which follows.

Note that $S^*(T)$, which is generally given in terms of the stationary distribution of $R(t)$ by (2), is alternatively also given in terms of the distribution of E_s by (11) if Assumption 1 holds. Also, note that, as in the case of expression (2), if an integer S satisfies the condition in (11) with equality, i.e. if $F_{E_{S+1}}(T) = b/(h+b)$, then both S and $S + 1$ are optimal order-base-stock levels when the demand lead-time is T.

Proposition 2 leads to the following theorem.

Theorem 1. *Under Assumption 1, the optimal order-base-stock level $S^*(T)$ is piecewise constant and decreasing in the demand lead-time T. More specifically, there exist break points $T_0, T_1, \ldots, T_{N+1}$, where $T_0 = 0$, such that if $T_n \le T < T_{n+1}$, then $S^*(T) = N - n$, $n = 0, \ldots, N$, where N is the optimal order-base-stock level when $T = T_0 = 0$, i.e., when there is no ADI. This means that for each break point T_n, $n = 1, \ldots, N + 1$, the following holds:*

$$F_{E_{N-n+q}}(T_n) > b/(h+b), \quad q = 2, \ldots, n+1,$$

$$F_{E_{N-n+1}}(T_n) \ge b/(h+b),$$

$$F_{E_{N-n-r}}(T_n) < b/(h+b), \quad r = 0, \ldots, N-n.$$

If $F_{E_{N-n}}(\cdot)$ is continuous, then the second inequality above is replaced by the equality

$$F_{E_{N-n+1}}(T_n) = b/(h+b).$$

In this case, $N - n$ and $N - n + 1$ are both optimal order-base-stock levels at $T = T_n$.

For the first point, T_0, which is equal to zero, the following holds:

$$F_{E_{N+1}}(T_0) \ge b/(h+b),$$

$$F_{E_{N-r}}(T_0) < b/(h+b), \quad r = 0, \ldots, N.$$

Finally, for values of T such that $T_n < T < T_{n+1}$, $n = 0, \ldots, N$, the following holds:

$$F_{E_{N-n+q}}(T) > b/(h+b), \quad q = 1, \ldots, n+1,$$

$$F_{E_{N-n-r}}(T) < b/(h+b), \quad r = 0, \ldots, N-n.$$

The proof of Theorem 1 is in the Appendix.

To better understand Theorem 1, we plotted the optimal order-base-stock level and the cumulative distribution function of E_n, $n = 0, \ldots, N + 1$, versus the demand lead-time in Figure 2. We assumed that the functions $F_{E_{N-n}}(\cdot)$ are continuous; therefore, according to Theorem 1, $F_{E_{N-n+1}}(T_n) = b/(h+b)$, $n = 1, \ldots, N+1$.

Theorem 1 states that if we increase the demand lead-time from one break point to the next, say from T_n to T_{n+1}, the optimal order-base-stock level drops by one unit, namely it drops from $N - n$ to $N - n - 1$. In other words, there is a tradeoff between the demand lead-time and the corresponding optimal order-base-stock level. This behavior is not surprising and has been noted in most of the related literature reviewed in Section 2. It stems from the fact that the larger the demand lead-time, the more additional time the supplier has to replenish his finished-goods inventory, and therefore the less finished-goods inventory he needs to hold.

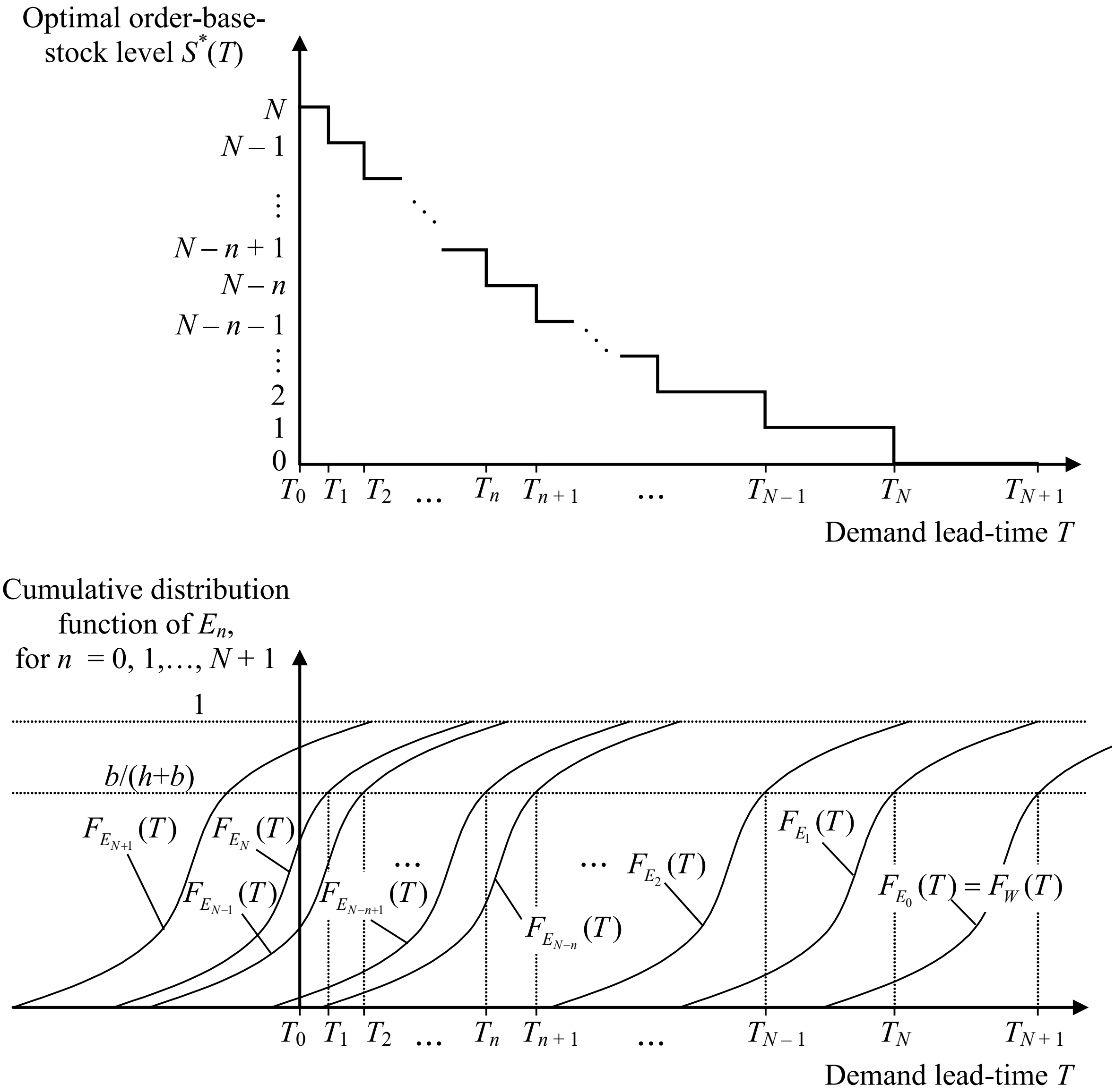

Figure 2. Optimal order-base-stock level versus demand lead-time, and cumulative distribution function of E_n, $n = 0,\dots, N+1$, versus demand lead-time.

The new insight that Theorem 1 brings to light is that under Assumption 1, this tradeoff is exhaustive, in the sense that the optimal order-base-stock level drops all the way to zero when the demand lead-time is sufficiently long. This means that the supplier can operate in all modes between pure make-to-stock and pure make-to-order, depending on the available amount of ADI.

More specifically, if the demand lead-time is zero, the supplier should operate in a pure make-to-stock mode with order-base-stock level $S^*(0) = N$. If the lead-time is equal to T_N, on the other hand, then the optimal order-base-stock level drops to zero, i.e. $S^*(T_N) = 0$, which means that the supplier should operate in a pure make-to-order mode, i.e. without keeping any finished-goods inventory. In all other intermediate cases, the supplier should operate in a mixed make-to-stock and make-to-order mode. Moreover, as the demand lead-time increases beyond T_N, the optimal order-base-stock level remains constant at zero, which means that the supplier should keep operating in a make-to-order mode.

It can be shown that as the demand lead-time T increases from 0 to T_{N+1}, the optimal long-run expected average cost of the supplier decreases as a result of the reduction in finished-goods holding costs. The minimum optimal long-run expected average cost is attained when $T = T_{N+1}$. For this reason, we refer to T_{N+1} as the *optimal* demand lead-time. Note that T_{N+1} is greater than T_N, which is the smallest demand lead-time for which the optimal order-base-stock level is zero. As T increases beyond T_{N+1}, however, the optimal long-run expected average cost increases, because the supplier replenishes his inventory too early with respect to his customer's due dates. To remedy this, the supplier could use a *modified* order-base-stock replenishment policy in which the placement of each replenishment order is offset by T_{N+1} from the due-date of the customer order that triggered it, as is done in an MRP system with planned supply lead-time T_{N+1}. It is easy to see that when $T > T_{N+1}$, the dynamic evolution of the supplier's finished-goods inventory/surplus under the modified order-base-stock replenishment policy is identical to that under the original order-base-stock policy when $T = T_{N+1}$. Hence, the long-run expected average cost under the modified order-base-stock replenishment policy when $T > T_{N+1}$, is equal to the minimum optimal long-run expected average cost under the standard order-base-stock policy, which is attained when $T = T_{N+1}$. The modified order-base-stock replenishment policy is discussed in Karaesmen et al. (2002).

The optimal order-base-stock level when the demand lead-time is zero, $S^*(0) = N$, and the optimal demand lead-time T_{N+1}, i.e., the two extreme points of the tradeoff curve between the demand lead-time and the optimal order-base-stock level shown in Figure 2, satisfy certain conditions which are given by the following two corollaries of Proposition 2, respectively.

Corollary 1. *Under Assumption 1, the optimal order-base-stock level when the demand lead-time is zero, N, is given by*

$$N = S^*(0) = \arg\min_{S:S\,\text{int}} \left\{ F_{E_{S+1}}(0) = P\{W_0 \le A_{S+1}\} \ge b/(h+b) \right\}. \tag{13}$$

Corollary 2. *Under Assumption 1, the optimal demand lead-time, T_{N+1}, is given by*

$$T_{N+1} = \arg\min_{T} \left\{ F_{E_0}(T) = P\{W_0 \le T\} = F_W(T) \ge b/(h+b) \right\}. \tag{14}$$

Note the resemblance between expressions (3) and (14). Expression (3) gives a condition for $S^*(0)$, i.e. N, expressed in terms of the stationary cumulative distribution of the *number* of in-process replenishment orders, R. Expression (14) gives the same condition for T_{N+1} expressed in terms of the stationary cumulative distribution of the flow *time* of in-process replenishment orders, W. Also note that expression (14) is identical to (12).

Corollaries 1 and 2 concern the end points of the tradeoff curve between the demand lead-time and the optimal order-base-stock level, shown on the top graph of Figure 2. A question that naturally arises next is what is the shape of this curve? To put this in more specific terms, let ΔT_n denote the difference between two successive break points T_n and T_{n+1}, i.e.

$$\Delta T_n = T_{n+1} - T_n,\ n = 0,\ldots, N. \tag{15}$$

The question then is what the behavior of ΔT_n versus n?

The managerial interest in this question stems from the fact that if, for example, ΔT_n turned out to be increasing in n, this would mean that the greater the demand lead-time is, the more additional demand lead-time the supplier would need in order to lower his optimal order-base-stock level by one unit. In this case, the supplier would have diminishing inventory savings as the demand lead-time increases.

The following corollary gives a necessary and sufficient condition that ΔT_n must satisfy.

Corollary 3. *Under Assumption 1 and assuming that the functions $F_{E_{N-n}}(\cdot)$, $n = 0,\ldots, N$, are continuous, the difference between successive break points, ΔT_n, satisfies*

$$F_{E_{N-n}}(T_n + \Delta T_n) = F_{E_{N-n}}(T_{n+1}) = F_{E_{N-n+1}}(T_n) = b/(h+b), \quad n = 1,\ldots, N. \tag{16}$$

From Corollary 3, it appears that ΔT_n generally depends on n. Proposition 3, which follows, gives a sufficient condition under which ΔT_n is independent of n.

Proposition 3. *Under Assumption 1 and assuming that the functions $F_{E_{N-n}}(\cdot)$, $n = 0,\ldots, N$, are continuous, if there exists a real number ΔT, $\Delta T > 0$, which satisfies*

$$P\{W \le T + \Delta T\} = P\{W \le T + H\} \tag{17}$$

then $\Delta T_n = \Delta T$, $n = 1,\ldots, N$.

The proof of Proposition 3 is in the Appendix.

Proposition 3 states that if there exists a real positive number ΔT, such that the probability that the replenishment time W is less than or equal to $T + \Delta T$ is equal to the probability that W is less than or equal to $T + H$, then the difference ΔT_n between any two successive break points T_n and T_{n+1}, $n = 1,\ldots, N$, is equal to ΔT and is therefore independent of n. This means that the additional demand lead-time that the supplier needs to have from his customers in order to lower his optimal order-base-stock level by one unit is constant and therefore independent of n. This further implies that the tradeoff between the order order-base-stock level and the demand lead-time is linear, and hence the supplier can achieve constant finished-goods inventory savings as the customer demand lead-time increases. Moreover, in this case, ΔT only depends on the distributions of the replenishment time W and the customer order interarrival time H, and not on the cost parameters h and b.

Proposition 3 leads to the following corollary.

Corollary 4. *If there exists a real positive number* ΔT *that satisfies* (17) *for any nonnegative real number* T*, then it must also satisfy* (17) *for* $T = 0$*, i.e. it must satisfy*

$$P\{W \le \Delta T\} = P\{W \le H\}\,. \tag{18}$$

Corollary 4 states that if ΔT satisfies (17), which according to Proposition 3 implies that $\Delta T_n = \Delta T$, $n = 1,\ldots, N$, and hence that the tradeoff between the optimal order-base-stock level and the demand lead-time is linear, then ΔT is such that the probability that the replenishment time W is less than or equal to ΔT is equal to the probability that W is less than or equal to the customer order interarrival time H.

To evaluate the usefulness of the results developed thus far, in the following section, we apply these results to three special cases for which we can obtain analytical results.

4 Application of Analysis to Special Cases

In this section, we focus our attention on the cases where the supply process is modeled as an M/D/1, M/M/1, and M/D/∞ queue, respectively. For the M/D/1 case, Karaesmen et al. (2003) presented some approximation- and simulation-based results. Our results in this section are exact. The M/M/1 and M/D/∞ cases were originally studied by Buzacott and Shanthikumar (1993, 1994) and Hariharan and Zipkin (1995), respectively, but we also look at them in this section in light of our analysis in the previous section. Note that, although in all cases Assumption 1 holds, in the M/D/1 and M/M/1 cases, the supply process is capacitated and sequential, whereas in the M/D/∞ case, it is uncapacitated and independent.

4.1 M/D/1 Supplier

For the case where the supply process is modeled as an M/D/1 queue, i.e. a queue with Poisson arrivals with mean arrival rate λ and a single server with deterministic service time equal to l, it is known from Queueing Theory that

$$\begin{aligned} &P_R(0) = 1-\rho, \\ &P_R(1) = (1-\rho)(e^{\rho}-1), \\ &P_R(n) = (1-\rho)\left[\sum_{k=1}^{n} e^{k\rho}(-1)^{n-k}\frac{(k\rho)^{n-k}}{(n-k)!} - \sum_{k=1}^{n-1} e^{k\rho}(-1)^{n-k-1}\frac{(k\rho)^{n-k-1}}{(n-k-1)!}\right], \quad n \geq 2, \end{aligned} \tag{19}$$

where $\rho = \lambda l$ (e.g., see Gross and Harris, 1998); therefore, $S^*(0)$ can be evaluated numerically from (3) as

$$S^*(0) = \arg\min_{S:S\,\mathrm{int}}\left\{\sum_{n=0}^{S} P_R(n) \geq b/(h+b)\right\}, \tag{20}$$

where $P_R(n)$ is given by (19).

In addition, the stationary probability function of $Z(t)$ is given by Proposition 4 that follows, where $a = T/l$ and $\lceil a \rceil$ denotes the smallest integer that is greater than or equal to a.

Proposition 4. *If the supply process is modeled as an M/D/1 queue, then the stationary probability function of $Z(t)$ for nonnegative values of $Z(t)$ is given by*

$$P_Z(n) = \left(\lceil a \rceil - a\right)P_R(n+\lceil a \rceil - 1) + \left(a+1-\lceil a \rceil\right)P_R(n+\lceil a \rceil), \quad n > 0. \tag{21}$$

The proof of Proposition 4 is in the Appendix.

Based on Proposition 4, the optimal order-base-stock level $S^*(T)$ and the break points T_n, $n = 1,\ldots, N$, can be evaluated from the following proposition.

Proposition 5. *If the supply process is modeled as an M/D/1 queue, then the optimal order-base-stock level $S^*(T)$ is given by*

$$S^*(T) = \arg\min_{S:S\,\mathrm{int}}\left\{\left(\lceil a \rceil - a\right)P_R(S+\lceil a \rceil) + 1 - F_R(S+\lceil a \rceil) \leq h/(h+b)\right\}. \tag{22}$$

In addition, break point T_n, $n = 1,\ldots, N+1$, satisfies

$$\left(\lceil T_n/l \rceil - T_n/l\right)P_R(N-n+\lceil T_n/l \rceil) + 1 - F_R(N-n+\lceil T_n/l \rceil) = h/(h+b). \tag{23}$$

The proof of Proposition 5 is in the Appendix.

Note that T_{N+1} can be obtained from (23) for $n = N + 1$, as follows

$$\left(\lceil T_{N+1}/l\rceil - T_{N+1}/l\right) P_R(\lceil T_{N+1}/l\rceil - 1) + 1 - F_R(\lceil T_{N+1}/l\rceil - 1) = h/(h+b). \quad (24)$$

Finally, Proposition 5 leads to the following proposition regarding the difference between any two successive break points.

Proposition 6. *If the supply process is modeled as an M/D/1 queue, then the difference* ΔT_n *between any two successive break points* T_n *and* T_{n+1}, $n = 1,\ldots, N$, *is independent of n and is equal to the deterministic service time l, i.e.*

$$\Delta T_n = \Delta T = l, \quad n = 1,\ldots, N. \quad (25)$$

The proof of Proposition 6 is in the Appendix.

Proposition 6 is surprisingly simple and very intuitive once it is stated. It says that if all customers provide the supplier with an additional demand lead-time which is equal to the deterministic service time, then the supplier can lower his optimal order-base-stock level by one unit. Intuitively, the reason he can do this is that during this additional customer demand lead-time he can produce exactly one unit. This result is also shown in Karaesmen et al. (2003), but there it is based on an approximation of the M/G/1 queue, whereas here it is based on exact analysis.

Moreover, Proposition 6 states that the difference between two successive break points only depends on – in fact, is equal to – the service time l and does not depend on the customer order arrival rate λ, and hence the load, ρ. The above observations suggest that Proposition 6 might possibly also hold for the more general case where the supplier is modeled as a G/D/1 queue.

A candidate alternative way to show that ΔT_n is constant and equal to l, for $n = 1,\ldots, N$, might have been to invoke Proposition 3. However, if we look more carefully at the "fine print" of Proposition 3, we will note that this is not possible, because Proposition 3 only holds for cases where the functions $F_{E_{N-n}}(\cdot)$, $n = 0,\ldots, N$, are continuous, whereas in the case of the M/D/1 queue, the distribution of E_0, which is equal to W, has a probability mass of $1 - \rho$ at $W = l$ and hence is not continuous. In fact, Proposition 7 that follows states that for the case where the supplier is modeled as an M/D/1 queue, equality (18) of Corollary 4 does not hold, implying that equality (17) of Proposition 3 does not hold either.

Proposition 7. *If the supply process is modeled as an M/D/1 queue, then*

$$P\{W \le \Delta T\} \ne P\{W \le H\}. \tag{26}$$

The proof of Proposition 7 is in the Appendix.

4.2 M/M/1 Supplier

For the case where the supply process is modeled as an M/M/1 queue, i.e. a queue with Poisson arrivals with mean arrival rate λ and a single exponential server with mean service rate μ, it is well-known from elementary Queueing Theory that $P_R(n) = (1 - \rho)\rho^n$ and hence $F_R(n) = 1 - \rho^{n+1}$, $n = 0, 1, \ldots$. It is also known that $F_W(t) = 1 - e^{-\mu(1-\rho)t}$, $t \ge 0$, where ρ is the system load and is given by $\rho = \lambda/\mu$ (e.g., see Gross and Harris, 1998); therefore, $S^*(0)$ can be evaluated from (3) as

$$\begin{aligned} S^*(0) &= \arg\min_{S:S\,\text{int}} \left\{ \rho^{S+1} \le h/(h+b) \right\} = \arg\min_{S:S\,\text{int}} \left\{ (S+1)\ln\rho \le \ln\left(h/(h+b)\right) \right\} \\ &= \left\lceil \left(\ln\left(h/(h+b)\right)/\ln\rho \right) - 1 \right\rceil. \end{aligned} \tag{27}$$

Alternatively, $S^*(0)$ can be evaluated from (13) as

$$\begin{aligned} S^*(0) &= \arg\min_{S:S\,\text{int}} \left\{ \int_0^\infty F_W(t) f_{A_{S+1}}(t)dt \ge b/(h+b) \right\} = \arg\min_{S:S\,\text{int}} \left\{ \int_0^\infty \left(1 - e^{-\mu(1-\rho)t}\right) \frac{\lambda e^{-\lambda t}(\lambda t)^S}{S!} dt \ge b/(h+b) \right\} \\ &= \arg\min_{S:S\,\text{int}} \left\{ 1 - \left(\lambda/\mu\right)^{S+1} \int_0^\infty \frac{\mu e^{-\mu t}(\mu t)^S}{S!} dt \ge b/(h+b) \right\} = \arg\min_{S:S\,\text{int}} \left\{ 1 - \rho^{S+1} \ge b/(h+b) \right\} \\ &= \left\lceil \left(\ln\left(h/(h+b)\right)/\ln\rho \right) - 1 \right\rceil, \end{aligned}$$

where we used the fact that A_{S+1} is Erlang distributed with shape parameter $S + 1$ and scale parameter λ, and the last equality of the above expression follows directly from (27).

From (4), Buzacott and Shanthikumar (1993, 1994) show that the stationary probability distribution function of $Z(t)$ for positive values of $Z(t)$ is given by

$$P_Z(n) = e^{-\mu(1-\rho)T} P_R(n) = \left(1 - F_W(T)\right) P_R(n), \quad n > 0. \tag{28}$$

The above expression implies that

$$\overline{F}_Z(n) = e^{-\mu(1-\rho)T} \overline{F}_R(n), \quad n > 0. \tag{29}$$

With this in mind, $S^*(T)$ can be evaluated from (2) as

$$S^*(T) = \arg\min_{S:S\,\text{int}}\left\{1-\left(e^{-\mu(1-\rho)T}\overline{F}_R(S)\right) \geq b/(h+b)\right\} = \arg\min_{S:S\,\text{int}}\left\{\rho^{S+1} \leq h/(h+b)e^{\mu(1-\rho)T}\right\}$$
$$= \arg\min_{S:S\,\text{int}}\left\{(S+1) \geq \left(\ln\left(h/(h+b)\right)/\ln\rho\right)+\left(\mu(1-\rho)/\ln\rho\right)T\right\} \quad (30)$$
$$= \left\lceil \left(\ln\left(h/(h+b)\right)/\ln\rho\right)+\left(\mu(1-\rho)/\ln\rho\right)T-1\right\rceil.$$

In addition, T_{N+1} can be evaluated from (14) as

$$T_{N+1} = \arg\min_{T}\left\{1-e^{-\mu(1-\rho)T} \geq b/(h+b)\right\} = \arg\min_{T}\left\{-\mu(1-\rho)T \leq \ln\left(h/(h+b)\right)\right\}$$
$$= -\ln\left(h/(h+b)\right)/\mu(1-\rho). \quad (31)$$

Finally, the difference between any two successive break points is given by the following proposition.

Proposition 8. *If the supply process is modeled as an M/M/1 queue, then the difference* ΔT_n *between any two successive break points* T_n *and* T_{n+1}*,* $n = 1,\ldots, N$*, is independent of n and is constant, i.e.,* $\Delta T_n = \Delta T$*, where*

$$\Delta T = -\ln\rho/\left(\mu(1-\rho)\right). \quad (32)$$

A proof of Proposition 8, based on Proposition 3, can be found in the Appendix. Alternatively, Proposition 8 can be shown by observing from (30) that $S^*(T)$ is piece-wise constant with break point spaced $-\ln\rho/\left(\mu(1-\rho)\right)$ time units apart.

Proposition 8 states that if the supply process is modeled as an M/M/1 queue, the difference between two successive break points, ΔT_n, is independent of n and is in fact constant. This means that the supplier has constant inventory savings as the demand lead-time increases. This constant is equal to the mean replenishment time L, which is equal to $1/(\mu(1-\rho))$, multiplied by $-\ln(\rho)$. In other words, the difference between two successive break points depends on both the mean service rate μ as well as the load ρ, but does not depend on the cost coefficients h and b.

4.3 M/D/∞ Supplier

For the case where the supply process is modeled as an M/D/∞ queue, i.e. a queue with Poisson arrivals with mean arrival rate λ, and independent, deterministic replenishment times equal to L, i.e. $W = L$, it is well known from elementary Queueing Theory that the stationary distribution or $R(t)$ is Poisson with mean $\rho = \lambda L$, i.e., $P_R(n) = e^{-\rho}\rho^n/n!$, $n = 0, 1, \ldots$ It is also easy to see that $F_W(t) = 0$ or 1, if $t < L$ or $t \geq L$, respectively.

With this in mind, $S^*(0)$ can be evaluated numerically from (3) as

$$S^*(0) = \arg\min_{S:S\,\text{int}} \left\{ e^{-\rho} \sum_{n=0}^{S} \rho^n \big/ n! \geq b/(h+b) \right\}. \tag{33}$$

Moreover, it is easy to see that any system with independent, deterministic replenishment times L and customer demand lead-times T, such that $T \leq L$, is equivalent to a system with independent, deterministic replenishment times $L - T$ and zero customer demand lead-times, as was first noted in Hariharan and Zipkin (1995). In other words, increasing T by an amount has exactly the same effect as decreasing L by the same amount. This implies that

$$P_Z(n) = e^{-\rho'} \rho'^n \big/ n!, \; n = 0, 1, \ldots,$$

where $\rho' = \lambda(L - T)$.

With this in mind, $S^*(T)$ can be evaluated numerically from (3) as

$$S^*(T) = \arg\min_{S:S\,\text{int}} \left\{ e^{-\rho'} \sum_{n=0}^{S} \rho'^n \big/ n! \geq b/(h+b) \right\}. \tag{34}$$

Alternatively, $S^*(T)$ can be evaluated from (11) as

$$S^*(T) = \arg\min_{S:S\,\text{int}} \left\{ P\{W - A_{S+1} \leq T\} \geq b/(h+b) \right\} = \arg\min_{S:S\,\text{int}} \left\{ 1 - F_{A_{S+1}}(L-T) \geq b/(h+b) \right\},$$

which would lead to exactly the same expression as (34), once we note that A_n is Erlang distributed with shape parameter n and scale parameter λ; hence, its cumulative distribution function is given by (e.g. see Buzacott and Shanthikumar 1993)

$$F_{A_n}(t) = 1 - \sum_{i=0}^{n-1} e^{-\lambda t} (\lambda t)^i \big/ i! \quad t > 0. \tag{35}$$

In addition, T_{N+1} can be easily obtained from (14) as

$$T_{N+1} = L, \tag{36}$$

after noting that $F_W(T) = 1$, if $T \geq L$, and 0, if $T < L$.

The managerial implication of expression (36) is simple. If all customers provide the supplier with a demand lead-time T which is equal to the deterministic replenishment time L, then the supplier does not need to keep any finished-goods inventory at all, because he can produce all the items requested during the demand lead-time. This means that he never has any backorders either; therefore his long-run expected average cost is zero. This is the ideal situation for the supplier.

Moreover, T_N can be determined from equation (34) as follows

$$\begin{aligned} T_N &= \arg\min_{T} \left\{ S^*(T) = 0 \right\} = \arg_{T} \left\{ e^{-\lambda(L-T)} = b/(h+b) \right\} = \arg_{T} \left\{ -\lambda(L-T) = \ln\left(b/(h+b)\right) \right\} \\ &= L - \frac{-\ln\left(b/(h+b)\right)}{\lambda}. \end{aligned} \tag{37}$$

Expression (37) states that the optimal order-base-stock level of the supplier is zero even if T falls short of L by $\ln(b/(h + b))/\lambda$ time units.

Although increasing T by an amount has exactly the same effect as decreasing L by the same amount, which makes ρ' a linear function of T, the tradeoff between the demand lead-time and the order-base-stock level is not linear, as was the case with the M/D/1 and M/M/1 systems. This is because the difference between any two successive break points is not constant, as the following proposition states.

Proposition 9. *If the supply process is modeled as an M/D/∞ queue, then the difference ΔT_n between any two successive break points T_n and T_{n+1}, n = 1,…, N, is given by*

$$\Delta T_n = F_{A_{N-n+1}}{}^{-1}(h/(h+b)) - F_{A_{N-n}}{}^{-1}(h/(h+b)), \tag{38}$$

where $F_{A_0}{}^{-1}(\cdot) = 0$, *and* $F_{A_n}{}^{-1}(u)$, n = 1,…, N, *is defined to equal that value of t for which* $F_{A_n}(t) = u$, *where* $F_{A_n}(t)$ *is given by* (35).

The proof of Proposition 9 is in the Appendix.

Proposition 9 implies that if the supply process is modeled as an M/D/∞ queue, ΔT_n depends on n, λ, h, and b, but does not depend on the deterministic replenishment time L. Since expression (38) is not in closed form, however, it is difficult to characterize the shape of this dependence. We will elaborate more on this issue in Section 5, where we will solve a numerical example.

5 Numerical Example

In this section, we illustrate the results derived for the three cases in the previous section with a numerical example.

First, consider the case where the supplier is modeled as an M/M/1 queue with mean arrival rate $\lambda = 0.8$ and mean service rate $\mu = 1.0$; hence the system load is $\rho = 0.8/1.0 = 0.8$ and the mean replenishment time is equal to $L = 1/(1.0(1 - 0.8)) = 5.0$. Suppose that the unit costs per unit time of holding finished-goods inventory and customer backorders are $h = 1$ and $b = 9$, respectively. Then, the optimal order-base-stock level when the demand lead-time is zero can be computed from (27) as $S^*(0) = \lceil \ln(1/(1 + 9))/\ln 0.8 - 1 \rceil = \lceil 9.3188 \rceil = 10$. Moreover, the optimal demand lead-time can be computed from (31) as $T_{N+1} = -\ln(1/(1 +$

9))/(1.0(1 – 0.8)) = 11.5129. Finally, the difference between any two successive break points can be computed from (32) as $\Delta T = -\ln 0.8/(1.0(1 - 0.8)) = 1.1157$.

Next, consider the case where the supplier is modeled as an M/D/1 queue with mean arrival rate $\lambda = 0.8$ and deterministic service times l. Suppose that l is such that the mean replenishment time of the M/D/1 system is equal to that of the M/M/1 system, i.e. $L = 5.0$, in order for the comparison between the two systems to be fair. It is well-known from the Pollaczek-Khintchine formula in Queueing Theory that the mean waiting (replenishment) time in the M/D/1 queue is given by $L = l + \rho^2/(2(1 - \rho)\lambda)$, where $\rho = \lambda l$ (e.g., see Gross and Harris, 1998). Solving this equation for l, with $L = 5$, yields $l = 1.0961$, and therefore $\rho = (0.8)(1.0961) = 0.8769$. As is expected, the utilization in the M/D/1 case is much higher than in the M/M/1 case. Suppose that $h = 1$ and $b = 9$, as in the M/M/1 system. Then, the optimal order-base-stock level when the demand lead-time is zero can be computed numerically from (20). The result is $S^*(0) = 9$. Moreover, the optimal demand lead-time can be computed numerically from (24). The result is $T_{N+1} = 10.5757$. Finally, the difference between any two successive break points can be computed from (25) as $\Delta T = 1.0961$.

Interestingly, the results for the M/M/1 and the M/D/1 cases are not too different. More specifically, $S^*(0)$ is bigger by just one unit in the M/M/1 case, whereas ΔT differs by less than two percent between the two cases. If we change the mean arrival rate from $\lambda = 0.8$ to $\lambda = 0.9$, so that the system load in the M/M/1 system is $\rho = 0.9/1.0 = 0.9$, and the mean replenishment time is $L = 1/(1.0(1 - 0.9)) = 10.0$, and repeat all the calculations, we will find that $S^*(0) = 21$, $T_{N+1} = 23.0258$, and $\Delta T = 1.0536$, for the M/M/1 case. The corresponding results for the M/D/1 case are $S^*(0) = 21$, $T_{N+1} = 21.0608$, and $\Delta T = 1.0496$, where $l = 1.0496$ and therefore $\rho = (0.95)(1.0496) = 0.9446$ so that $L = 10.0$. Again, the results between the two cases are quite close.

Finally, suppose that the supplier is modeled as an M/D/∞ queue with the same mean arrival rate as in the M/M/1 system, i.e. $\lambda = 0.8$, and deterministic replenishment times that are equal to the mean replenishment time in the M/M/1 system, i.e. $L = 5.0$; therefore $\rho = (0.8)(5.0) = 4.0$. Suppose that $h = 1$ and $b = 9$, as in the M/M/1 and M/D/1 systems. Then, the optimal order-base-stock level when the demand lead-time is zero can be computed numerically from (33). The result is $S^*(0) = 7$. Moreover, the optimal demand lead-time can be computed from (36) as $T_{N+1} = L = 5$. Finally, the difference between any two successive break points can be computed numerically from (38). Once this is done, the break points themselves can be easily computed. The results are shown in Table 1.

Table 1: Break points and their differences for the M/D/∞ queue example for the case where $h = 1$ and $b = 9$

n	$[T_n, T_{n+1}]$	$\Delta T_n = T_{n+1} - T_n$	$S^*(T)$, $T \in [T_n, T_{n+1}]$
0	[0.0, 0.1315]	0.1315	7
1	[0.1315, 1.0601]	0.9286	6
2	[1.0601, 1.9593]	0.8991	5
3	[1.9593, 2.8190]	0.8598	4
4	[2.8190, 3.6224]	0.8034	3
5	[3.6224, 4.3352]	0.7128	2
6	[4.3352, 4.8683]	0.5331	1
7	[4.8683, 5.0]	0.1317	0

From Table 1, we can observe that the difference between successive break points ΔT_n, $n = 1, \ldots, N$, is decreasing in n; in other words, the tradeoff between the optimal order-base-stock level and the demand lead-time is concave. This means that the greater the demand lead-time is, the less the additional order lead that is needed in order for the supplier to lower his optimal order-base-stock level by one unit.

This observed behavior was not obvious from the beginning. It implies that the supplier has increasing inventory savings as the demand lead-time increases. For example, when the demand lead-time is small, say $T_1 = 0.1315$, and the optimal order-base-stock level is 6, if the supplier wants to reduce his optimal order-base-stock level, from 6 to 5, he needs to have his customers increase their demand lead-time by 0.9286 to $T_2 = 1.0601$. On the other hand, when the demand lead-time is large, say $T_6 = 4.3352$, and the optimal order-base-stock level is 1, if the supplier wants to reduce his optimal order-base-stock level, from 1 to 0, he needs to have his customers increase their demand lead-time by only 0.5331 to $T_7 = 4.8683$. This should be good news for the supplier, especially because the cost associated with obtaining ADI is certainly increasing – probably with an augmenting rate – in the customer demand lead-time.

The above observed behavior, however, only holds for low values of $h/(h + b)$. Table 2 shows the differences between successive break points for several different values of $h/(h + b)$. These differences were computed numerically from (38).

From Table 2, we can see that when $h/(h + b)$ is equal to 0.1, 0.3 or 0.5, the difference between successive break points ΔT_n, $n = 1, \ldots, N$, is decreasing in n. When $h/(h + b)$ is equal to 0.7 or 0.9, on the other hand, this is no longer true. The reason for this diversity in the behavior of ΔT_n vs. n is hidden in the shape of $F_{A_n}(t)$, given by (35), which is sigmoidal with exactly one inflection point at $t = (n - 1)/\lambda$. We say "hidden," because it is very difficult to provide exact analytical conditions under which ΔT_n is either decreasing or increasing in n, as ΔT_n is given by expression (38) which involves the inverse of $F_{A_{N-n}}(t)$ and $F_{A_{N-n+1}}(t)$.

Table 2: Difference between break points for the M/D/∞ queue example for different values of $h/(h + b)$

	$h/(h+b)$				
	0.1	0.3	0.5	0.7	0.9
ΔT_0	0.1315	0.4580	0.4099	0.4805	0.1379
ΔT_1	0.9286	1.0874	1.2475	1.4704	1.9839
ΔT_2	0.8991	1.0624	1.2447	1.5441	2.8782
ΔT_3	0.8598	1.0205	1.2315	1.5050	
ΔT_4	0.8034	0.9258	0.8664		
ΔT_5	0.7128	0.4458			
ΔT_6	0.5331				
ΔT_7	0.1317				

6 Conclusions

The first important finding in this paper is that if Assumption 1 holds, i.e. if the supplier's replenishment orders arrive in the order that they are placed, then the tradeoff between the optimal order-base-stock level and the demand lead-time is exhaustive, in the sense that the optimal order-base-stock level drops all the way to zero if the demand lead-time is sufficiently long. In most manufacturing systems, replenishment (production) times are more or less sequential, so Assumption 1 holds naturally. Assumption 1 should also hold for many pure inventory systems, for which replenishment times are not really independent, especially if the supplier's replenishment orders come for a single vendor. An open question that arises naturally is what happens if Assumption 1 does not hold? An example where Assumption 1 does not hold is the case where our supplier is modeled as an M/G/∞ queueing system. Such a model might be valid in practice if, for example, the supplier's replenishment orders came for multiple vendors.

Numerical evidence in Liberopoulos et al. (2003) suggests that if Assumption 1 does not hold, i.e. if the supplier's replenishment orders do not arrive in the order that they are placed, then the tradeoff between the optimal order-base-stock level and the demand lead-time is not exhaustive. Instead, in that case, as the demand lead-time increases, the optimal order-base-stock level decreases, as in the case in which Assumption 1 does hold, but only until it reaches a certain minimum positive level at a critical demand lead-time value. As the demand lead-time increases beyond this critical value, the optimal order-base-stock level remains fixed at the minimum level, while the replenishment orders should be delayed by the demand lead-time offset by the critical value. The numerical evidence in Liberopoulos et al. (2003) further suggests that this critical value is equal to or close to the average replenishment lead-time, but there is no proof of this to date.

Another important finding in this paper is that for the case where the supplier is modeled as an M/D/1 queueing system, the tradeoff between the optimal order-base-stock level and the demand lead-time is linear, just as in the case where the supplier is modeled as an M/M/1 queueing system. More specifically, it was shown that for the M/D/1 case, the optimal order-base-stock level decreases by one unit, if the demand lead-time increases by an amount equal to the supplier's constant processing time. As was mentioned earlier, intuitively, the reason he can do this is that during this additional customer demand lead-time he can produce exactly one unit. Given that the tradeoff is linear for the M/D/1 and M/M/1 cases, an open question that arises naturally is whether this tradeoff remains linear for the general case where the supplier is modeled as an M/G/1 queueing system. The approximate analysis of the M/G/1 case and the numerical experiments in Karaesmen et al. (2003) suggest that it does, but there is no proof of this to date. If a proof that the tradeoff is linear for the M/G/1 could be constructed, a further question to ask is whether it is also linear for the G/G/1 case or more generally for any case where the supply process is strictly sequential.

Another open question which is related to the linearity of the tradeoff is whether the condition of Proposition 3 is not only sufficient but is also necessary.

The last finding in this paper is that for the case where the supplier is modeled as an M/D/∞ queueing system, the tradeoff between the optimal order-base-stock level and the demand lead-time is sometimes concave and sometimes not, depending on the cost ratio *h/b* and other problem parameters. An open question that arises naturally is what are the conditions under which this tradeoff is concave or non-concave? As was mentioned earlier, however, answering this question may prove to be a formidable task, because ΔT_n is given implicitly by expression (38) which involves the inverse of $F_{A_n}(\cdot)$.

Appendix

Proof of Theorem 1.

Proposition 2 implies that the optimal order-base-stock level for a given demand lead-time T, $S^*(T)$, satisfies $F_{E_{S^*(T)+1}}(T) \geq b/(h+b)$ and $F_{E_{S^*(T)-r}}(T) < b/(h+b)$, $r = 0,\ldots, S^*(T)$. Suppose that we increase the demand lead-time from T to $T + \Delta T$, where $\Delta T > 0$. According to (9) and (10), the functions $F_{E_{S^*(T)-r}}(T+\Delta T)$, $r = 0,\ldots, S^*(T)$, are increasing in ΔT, with $F_{E_{S^*(T)-r}}(T+\Delta T) > F_{E_{S^*(T)-r-1}}(T+\Delta T)$, $r = 0,\ldots, S^*(T) - 1$. This implies that there a exists a finite value of ΔT, say $\Delta T'$, such that $F_{E_{S^*(T)}}(T+\Delta T') \geq b/(h+b)$ and $F_{E_{S^*(T)-r}}(T+\Delta T') < b/(h+b)$, $r = 1,\ldots, S^*(T)$. From Proposition 2, this means that if the demand lead-time reaches $T + \Delta T'$, the optimal order-base-stock level drops from $S^*(T)$ to $S^*(T) - 1$, i.e., $S^*(T + \Delta T') = S^*(T) - 1$.

Suppose that we further increase the demand lead-time beyond $T + \Delta T'$. Then, following the same argument, there exists another value of ΔT, say $\Delta T''$, where $\Delta T'' > \Delta T'$, such that $F_{E_{S^*(T)-1}}(T+\Delta T'') \geq b/(h+b)$ and $F_{E_{S^*(T)-r}}(T+\Delta T'') < b/(h+b)$, $r = 2,\ldots, S^*(T)$. Again, from Proposition 2, this means if the demand lead-time reaches $T + \Delta T''$, the optimal order-base-stock level further drops from $S^*(T) - 1$ to $S^*(T) - 2$, i.e., $S^*(T + \Delta T'') = S^*(T) - 2$.

Applying the above argument repeatedly leads to Theorem 1. □

Proof of Proposition 3.

The cumulative distribution of E_{N-n} at $z + \Delta T$, $z \geq 0$, $\Delta T > 0$, is given by

$$F_{E_{N-n}}(z+\Delta T) = P\{W_0 - A_{N-n} \leq z + \Delta T\} = \int_0^\infty F_W(z+\Delta T + x) f_{A_{N-n}}(x)dx. \tag{39}$$

Similarly, the cumulative distribution of E_{N-n+1} at z, $z \geq 0$, is given by

$$\begin{aligned} F_{E_{N-n+1}}(z) &= \int_0^\infty F_W(z+y) f_{A_{N-n+1}}(y)dy = \int_0^\infty F_W(z+y)\left(\int_0^y f_H(y-x) f_{A_{N-n}}(x)dx\right)dy \\ &= \int_0^\infty \left(\int_0^\infty F_W(z+x+t) f_H(t)dt\right) f_{A_{N-n}}(x)dx. \end{aligned} \tag{40}$$

Now, suppose that the following equality holds:

$$F_W(z+x+\Delta T) = \int_0^\infty F_W(z+x+t) f_H(t)dt, \quad x \geq 0, \quad z \geq 0. \tag{41}$$

If we perform a change of variables from x to T, where $T = z + x$, equality (41) can be rewritten as

$$F_W(T+\Delta T) = \int_0^\infty F_W(T+t) f_H(t)dt, \quad T \geq z, \quad z \geq 0. \tag{42}$$

Equality (41) and therefore also equality (42) implies that the right-hand sides of equalities (39) and (40) are equal to each other, which further implies that that left-hand sides are also equal to each other, i.e.

$$F_{E_{N-n}}(z+\Delta T) = F_{E_{N-n+1}}(z), \quad z \geq 0. \tag{43}$$

To prove Proposition 3, we need to show that equality (17) implies that

$$F_{E_{N-n}}(T_n+\Delta T) = F_{E_{N-n+1}}(T_n), \tag{44}$$

which, from (16), implies that $\Delta T_n = \Delta T$. Indeed, equality (17) can be written as equality (42), which as was mentioned above implies equality (43). However, equality (43) also implies equality (44) when $z = T_n$, and the proof is complete. □

Proof of Proposition 4.

For the M/D/1 queue, in order to find $P_{G_T^k}(i)$, $i < k$, $k \geq 1$, which is needed to compute $P_Z(n)$, $n > 0$, from expression of (4), we note the following. Given that at the beginning of a time interval of length T there are k in-process replenishment orders, the number of replenishment completions, i, in this time interval, where $i < k$ and $k \geq 1$, is $\lceil a \rceil - 1$, if the percentage of the remaining time to completion of the in-process replenishment order in service at the beginning of the time interval is strictly greater than $\lceil a \rceil - a$, and $\lceil a \rceil$, if it is smaller than or equal to $\lceil a \rceil - a$.

Given that the service time is deterministic and equal to l and that the arrival process is Poisson, the remaining time in service of the in-process replenishment order in service at the beginning of the time interval is a random variable which is uniformly distributed over the interval [0, l]; therefore, the probability that the percentage of the remaining time to completion is greater than $\lceil a \rceil - a$ is equal to $\lceil a \rceil - a$. Similarly, the probability that it is smaller than or equal to $\lceil a \rceil - a$ is equal to $1 - (\lceil a \rceil - a) = a + 1 - \lceil a \rceil$. This means that

$$P_{G_T^k}(i) = \begin{cases} \lceil a \rceil - a, & i = \lceil a \rceil - 1, \\ a + 1 - \lceil a \rceil, & i = \lceil a \rceil, \\ 0, & \text{otherwise.} \end{cases}$$

If we substitute $P_{G_T^k}(k-n)$ from the above expression into (4) for $n > 0$ we obtain an expression that only has two non-zero terms. This expression is (21). □

Proof of Proposition 5.

Based on Proposition 4, the optimal order-base-stock level $S^*(T)$ for the M/D/1 queue can be derived from expression (2) as follows:

$$S^*(T) = \arg\min_{S:S\,\mathrm{int}}\left\{ \sum_{n=S+1}^{\infty} P_Z(n) \le h/(h+b) \right\}$$
$$= \arg\min_{S:S\,\mathrm{int}}\left\{ \sum_{n=S+1}^{\infty} \left(\lceil a\rceil - a\right) P_R(n+\lceil a\rceil - 1) + \left(a+1-\lceil a\rceil\right) P_R(n+\lceil a\rceil) \le h/(h+b) \right\}$$
$$= \arg\min_{S:S\,\mathrm{int}}\left\{ \left(\lceil a\rceil - a\right) P_R(S+\lceil a\rceil) + 1 - F_R(S+\lceil a\rceil) \le h/(h+b) \right\}.$$

Setting $S^*(T) = N - n$ in the above expression implies that

$$\left(\lceil T/l\rceil - T/l\right) P_R(N-n+\lceil T/l\rceil) + 1 - F_R(N-n+\lceil T/l\rceil) \le h/(h+b), \tag{45}$$

where we have replaced a by T/l. From Theorem 1, break point T_n is the smallest demand lead-time T for which $S^*(T) = N - n$, i.e., which satisfies inequality (45). It is easy to see that as T decreases, the left-hand-side of inequality (45) increases continually. The smallest value of T for which inequality (45) holds, i.e. T_n, is that value of T which makes the left-hand-side of inequality (45) equal to its right-hand-side; therefore, T_n satisfies (23), and this completes the proof. □

Proof of Proposition 6.

For the M/D/1 queue, Proposition 5 implies that break point T_{n+1} must satisfy

$$\left(\lceil T_{n+1}/l\rceil - T_{n+1}/l\right) P_R(N-n-1+\lceil T_{n+1}/l\rceil) + 1 - F_R(N-n-1+\lceil T_{n+1}/l\rceil) = h/(h+b). \tag{46}$$

To prove Proposition 6, we must show that if T_n satisfies (23) and $\Delta T_n = l$, then the resulting value of T_{n+1}, which by definition is given by $T_{n+1} = T_n + \Delta T_n = T_n + l$, satisfies (46).

If we replace T_{n+1} by $T_n + l$ in the left-hand-side of (46), we obtain

$$\left(\lceil (T_n+l)/l\rceil - (T_n+l)/l\right) P_R(N-n-1+\lceil (T_n+l)/l\rceil) + 1 - F_R(N-n-1+\lceil (T_n+l)/l\rceil).$$

After some simplifications, the above expression can be shown to be identical to the left-hand-side of equation (23) which, as we have assumed, is equal to $h/(h + b)$. Therefore, if T_{n+1} is equal to $T_n + l$, equality (46) is satisfied, and the proof is complete. □

Proof of Proposition 7.

For the M/D/1 queue, the left-hand side of equality (18) can be written as

$$P\{W \le \Delta T\} = P\{W \le l\} = P\{W = l\} = 1-\rho. \tag{47}$$

Let Q denote the waiting time in queue of a replenishment order, and let R denote the residual service time of a replenishment order, i.e. the remaining service time of the replenishment order being served at the instant a new replenishment order arrives. A standard result in Queueing Theory is that for an M/G/1 queue, the distribution of Q is given by the following expression (e.g. see Gross and Harris, 1998):

$$F_Q(t) = (1-\rho)\sum_{n=0}^{\infty} \rho^n F_R^{(n)}(t),$$

where $F_R^{(n)}(t)$ is the nth convolution of $F_R(t)$. With this in mind and noting that $Q = W - l$, the right-hand side of equality (18) can be written as

$$\begin{aligned} P\{W \le H\} &= P\{Q + l \le H\} = \int_0^{\infty} F_Q(t) f_H(t+l)dt = (1-\rho)e^{-\rho}\int_0^{\infty}\left[\sum_{n=0}^{\infty}\rho^n F_R^{(n)}(t)\right]\lambda e^{-\lambda t}dt \\ &= (1-\rho)e^{-\rho}\sum_{n=0}^{\infty}\rho^n \int_0^{\infty} F_R^{(n)}(t)\lambda e^{-\lambda t}dt = (1-\rho)e^{-\rho}\sum_{n=0}^{\infty}\rho^n P\{R^{(n)} \le H\}, \end{aligned} \tag{48}$$

where $R(n)$ denote the sum of n i.i.d. random variables, each being distributed as R, where in the case of the M/D/1 queue, R is uniformly distributed between 0 and l.

Clearly, in order for the right-hand sides of equations (47) and (48) to be equal to each other, which would imply (18), the summation in the last equality of expression (48) should be equal to e^{ρ}. In order for this to happen, the term $P\{R(n) \le H\}$ inside the summation should be equal to $1/n!$, which is hardly the case. Therefore, the right-hand sides of equations (47) and (48) are not equal to each other, which implies (26). □

Proof of Proposition 8.

For the M/M/1 queue, the following two expressions hold:

$$P\{W \le T + \Delta T\} = 1 - e^{-\mu(1-\rho)(T+\Delta T)} = 1 - e^{-\mu(1-\rho)\Delta T} e^{-\mu(1-\rho)T} \tag{49}$$

$$\begin{aligned} \underset{H}{E}[P\{W \le T + H\}] &= \int_0^{\infty} F_W(T+t) f_H(t)dt = \int_0^{\infty}\left(1 - e^{-\mu(1-\rho)(T+t)}\right)\lambda e^{-\lambda t}dt \\ &= 1 - \rho e^{-\mu(1-\rho)T}\int_0^{\infty}\mu e^{-\mu t}dt = 1 - \rho e^{-\mu(1-\rho)T}. \end{aligned} \tag{50}$$

If ΔT is given by expression (32), then the right-hand sides of equations (49) and (50) above are equal to each other, and hence the left-hand sides are also equal to each other. By Proposition 3, this means that $\Delta T_n = \Delta T$, $n = 1,\dots, N$, and this completes the proof. □

Proof of Proposition 9.

For the M/D/∞ queue, the following holds:

$$F_{E_{N-n}}(T_{n+1}) = P\{W - A_{N-n} \le T_{n+1}\} = P\{A_{N-n} \ge L - T_{n+1}\} = P\{A_{N-n} \ge T_{N+1} - T_{n+1}\}$$
$$= P\{A_{N-n} \ge \sum_{i=n+1}^{N} \Delta T_i\} = 1 - F_{A_{N-n}}(\sum_{i=n+1}^{N} \Delta T_i), \quad n = 0, \ldots, N-1. \tag{51}$$

The above expression and equation (16) imply that

$$\sum_{i=n+1}^{N} \Delta T_i = F_{A_{N-n}}^{-1}(h/(h+b)).$$

This means that ΔT_n, $n = 1, \ldots, N-1$, can be computed recursively as follows:

$$\Delta T_N = F_{A_1}^{-1}(h/(h+b))$$
$$\Delta T_{N-1} = F_{A_2}^{-1}(h/(h+b)) - \Delta T_{N-1} = F_{A_2}^{-1}(h/(h+b)) - F_{A_1}^{-1}(h/(h+b))$$
$$\Delta T_{N-2} = F_{A_3}^{-1}(h/(h+b)) - \Delta T_{N-2} - \Delta T_{N-1} = F_{A_3}^{-1}(h/(h+b)) - F_{A_2}^{-1}(h/(h+b))$$
$$\vdots$$
$$\Delta T_n = F_{A_{N-n+1}}^{-1}(h/(h+b)) - F_{A_{N-n}}^{-1}(h/(h+b)),$$
$$\vdots$$
$$\Delta T_1 = F_{A_N}^{-1}(h/(h+b)) - F_{A_{N-1}}^{-1}(h/(h+b))$$

where $F_{A_n}(\cdot)$ is the cumulative distribution function of A_n and is given by (35). □

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