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Pretwisted beam subjected to thermal loads: A gradient thermoelastic analogue

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ABSTRACT

It is well known from the classical torsion theory that the cross section of a prismatic beam subjected to end torsional moments will rotate and warp in the longitudinal direction. Rotation is depicted through the angle of twist per unit length and depends in general on the position along the length of the beam, while the warping function addresses the longitudinal distortion of the unrotated cross sections. In the present study, we consider a prismatic beam that possesses an initial twist which is constant along its length. A thermal field is present along the beam and its ends are loaded with axial forces and torsional moments. The governing equilibrium equations and the corresponding boundary conditions were obtained using an energy variational statement. A one-dimensional gradient thermoelastic analogue is developed. The advantageous aspect of the present study is that the additional (and peculiar) boundary conditions required by the gradient elasticity theory and the related microstructural lengths, analogous to micromechanical lengths, emerge in a natural way from the geometrical characteristics of the beam cross section and the material properties. We have examined various examples with different cross sections and loads to demonstrate the applicability of the model to the design of special yarns useful in smart textiles and thermally activated microdrilling actuators.

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Introduction

Beams are load-carrying structural components used in many technological applications. Their prominent structural performance consists of conveying axial, bending, and torsional loads in a sufficient way. Simple beam theory dates back to the 17th century when Hooke stated his famous law and was established as a first-order approximation linear theory to investigate the response of a beam under tensile loads. Since then many research efforts have been devoted to enrich the classical beam theory driven by the ever demanding needs for more complicated structures performing to extreme loading excitations. Despite of the vast literature on the subject, even nowadays beam theory remains a versatile tool used in the analysis of very challenging and sophisticated problems in the area of mechanics.

In the aerospace industry, structures like helicopter blades, wind turbines, propellers, etc., can be modeled as simple beams supplemented with one additional characteristic, the *pretwist*. Pretwist brings into the analysis some complexity, especially due to the coupling of the various loading conditions. It is evident that a thorough investigation of pretwisted beams is not confined into a narrow academic research framework but extends beyond to provide solutions into real demanding structures. To this end, many fervent research efforts have been devoted to formulate a rigorous theoretical framework for

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beams accounting for all the complex aspects of the induced pretwist. It was no earlier than 50s, when Chu [1] showed that the torsional rigidity of thin-walled beams with elongated sections is increased with pretwist using an engineering approach. Over the same period of time, Okubo [2, 3] published independently his work on the helical springs and twisted beams by manipulating the three-dimensional equations of elasticity and formulating a two-dimensional boundary value problem. Later on, Rosen [4, 6, 7], Hodges [5], Shield [8], and Krenk [9] developed improved technical theories for pretwisted beams using kinematically admissible displacements and the theorem of minimum potential energy. Krenk and Gunneskov [10, 11], Kosmatka [12], Jiang and Henshall [13] tackled the problem through asymptotic analysis and the finite element method for cross sections of various shapes.

Aside the mechanical loads, the structural performance of pretwisted beams may be significantly affected by imposed thermal stresses. It is well known that the variation of a temperature field may induce thermal stresses in the elastic continuum which can be addressed through the constitutive law of the material. In the present study, it is assumed that the induced nonuniform temperature field is linear and our attention is confined to the case where quasi-static thermal conditions hold, meaning that the variation of the temperature field with time is slow. In this way, the fields of temperature and displacement can be thought of as totally decoupled. A thorough analysis regarding the various aspects of thermoelasticity can be found in many standard textbooks like Hetnarski and Eslami [14] and Boley and Weiner [15].

Classical continuum mechanics theory is often inadequate to describe the mechanical behavior of materials with microstructure due to the lack of length-scale parameters. Therefore, resort is often sought to more elaborate continuum theories where the role of the microstructure is involved through intrinsic parameters entering the constitutive law of the continuum. Toupin [16], Mindlin [17], Koiter [18], and Eringen [19] proposed generalized linear continuum theories which are characterised by stress nonlocality and the existence of material length scales. Mindlin's general theory [17] includes three equivalent forms defined on the basis of strain energy function expression of the continuum.

The present work deals with the problem of a pretwisted beam subjected to thermal loads. Infinitesimal strains and rotations are assumed throughout. A classical structural mechanics approach is used and an analogy with an one-dimensional strain gradient theory is presented. Mindlin's form II strain gradient elasticity theory is used and the strain elastic energy density function of the pretwisted beam is expressed in terms of the strain tensor and its second spatial gradients. The analogy with the gradient thermoelasticity stems from the coupling of the axial and torsional deformation, activated by temperature change. The results developed in the present article extend an earlier work of Kordolemis et al. [20]. The possible use of this coupling to the design of microdrilling actuators is discussed.

Problem formulation

Consider a homogeneous cylindrical beam with constant cross sections of arbitrary shape as shown in Figure 1. A fixed Cartesian coordinate system $Oxyz$ with base vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is introduced with the z-axis along the centroids of the cross sections and parallel to the generators of the cylinder. The beam is assumed to be of length L; one of its bases is on the xy-plane and the other is on the plane $z = L$. The beam is then pretwisted around the longitudinal z-axis by an amount of twist per unit length α_0, such that any cross section at a distance z from the origin rotates by an amount $\phi(z) = \alpha_0 z$. The pretwisted beam is assumed to be stress free. To ensure performance advantages, rotor blades of turbomachinery, tilt rotor aircraft, and helicopters are usually twisted. It is convenient to introduce a local Cartesian coordinate system $O\eta\zeta z$ on arbitrary cross section. The local coordinates (η, ζ) are related to the global coordinates (x, y) by the transformation formula

$$\eta\left(x, y, z\right) = x\cos(\alpha_0 z) + y\sin(\alpha_0 z) \tag{1}$$

$$\zeta\left(x, y, z\right) = -x\sin(\alpha_0 z) + y\cos(\alpha_0 z) \tag{2}$$

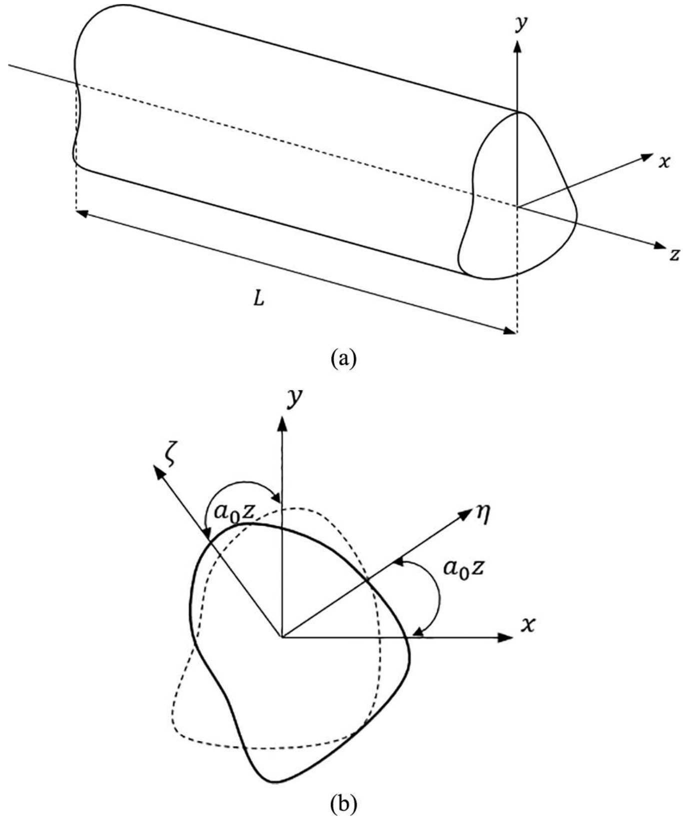


Figure 1. (a) The pretwisted beam with the local coordinate system and (b) the cross section of the beam.

so that

$$\frac{\partial \eta}{\partial z} = \alpha_0\, \zeta, \quad \frac{\partial \zeta}{\partial z} = -\alpha_0\, \eta, \quad \text{and} \quad \frac{\partial f\,(\eta, \zeta)}{\partial z} = \alpha_0 \left(\zeta \frac{\partial f}{\partial \eta} - \eta \frac{\partial f}{\partial \zeta} \right) \tag{3}$$

where $f\,(\eta, \zeta)$ is an arbitrary function.

The material of the beam is homogeneous, isotropic, and linearly elastic. The beam is loaded by axial forces and torsional moments. In addition to the mechanical loads, the beam is subjected to thermal loading which causes isotropic thermal expansion throughout the volume of the beam. The strains caused by the applied loads are assumed to be infinitesimal small so as the linear kinematics are applicable. The tensor of thermal strains $\boldsymbol{\varepsilon}^{\mathrm{th}}$ is purely volumetric and can be written in the form

$$\boldsymbol{\varepsilon}^{\mathrm{th}}(z) = \alpha\, \Delta\theta(z)\, \boldsymbol{\delta} \tag{4}$$

where α is the coefficient of thermal expansion, $\boldsymbol{\delta}$ the second-order identity tensor, and $\Delta\theta\,(z)$ the imposed (known) change of temperature.

Displacement field

We use an approximation for the displacement field in the beam of the general form (Krenk [9], Simo and Vu-Quoc [23])

$$\mathbf{u}(\mathbf{x}) = \mathbf{w}_0(z) + \boldsymbol{\phi}(z) \times \mathbf{p}(x, y) + \beta(z)\psi(\eta, \zeta)\, \mathbf{e}_z \tag{5}$$

where $\mathbf{w}_0(z)$ defines the displacement of the center of each cross section, $\mathbf{p}(x,y) = x\,\mathbf{e}_x + y\,\mathbf{e}_y$ is the position vector of material points on the cross section, $\boldsymbol{\phi}(z)$ is the infinitesimal rotation vector of the cross section, $\psi(\eta,\zeta)$ is the (known) Saint-Venant warping function of the cross section (i.e., the solution of the Saint-Venant torsion problem without pretwist, Sokolnikoff [21]), and $\beta(z)$ is the (unknown) warping amplitude. In particular, we use

$$\mathbf{w}_0(z) = w_1(z)\,\mathbf{e}_z, \quad \boldsymbol{\phi}(z) = \phi(z)\,\mathbf{e}_z, \quad \text{and} \quad \beta(z) = \frac{d\phi(z)}{dz} \tag{6}$$

This displacement field is defined completely by the generalized displacements $(w_1(z), \phi(z))$, which are determined by minimizing the corresponding potential energy of the beam. The corresponding stresses will satisfy approximately the equilibrium equations and the traction boundary conditions.

The components (u, v, w) in the (x, y, z) directions of the displacement field are[1]

$$u\left(y,z\right) = -\phi(z)\,y \tag{7}$$

$$v\,(x,z) = \phi(z)\,x \tag{8}$$

$$w\,(\eta,\zeta,z) = w_1\,(z) + \frac{d\phi\,(z)}{dz}\psi\,(\eta,\zeta) \tag{9}$$

The corresponding components of the infinitesimal mechanical (as opposed to thermal) strain tensor can be expressed as

$$\varepsilon_{xx}^{\mathrm{me}}(z) = \varepsilon_{xx} - \varepsilon_{xx}^{\mathrm{th}} = \frac{\partial u}{\partial x} - \alpha\,\Delta\theta = -\alpha\,\Delta\theta\,(z) \tag{10}$$

$$\varepsilon_{yy}^{\mathrm{me}}(z) = \varepsilon_{yy} - \varepsilon_{yy}^{\mathrm{th}} = \frac{\partial v}{\partial y} - \alpha\,\Delta\theta = -\alpha\,\Delta\theta\,(z) \tag{11}$$

$$\varepsilon_{zz}^{\mathrm{me}}(x,y,z) = \varepsilon_{zz} - \varepsilon_{zz}^{\mathrm{th}} = \frac{\partial w}{\partial z} - \alpha\,\Delta\theta = \frac{dw_1(z)}{dz} + \frac{d^2\phi\,(z)}{dz^2}\psi\,(\eta,\zeta) + \frac{d\phi\,(z)}{dz}\frac{\partial\psi\,(\eta,\zeta)}{\partial z} - \alpha\,\Delta\theta\,(z) \tag{12}$$

$$\varepsilon_{xy}^{\mathrm{me}} = \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0 \tag{13}$$

$$\varepsilon_{xz}^{\mathrm{me}}(x,y,z) = \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \frac{1}{2}\frac{d\phi\,(z)}{dz}\left[-y + \frac{\partial\psi\,(\eta,\zeta)}{\partial x}\right] \tag{14}$$

$$\varepsilon_{yz}^{\mathrm{me}}(x,y,z) = \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \frac{1}{2}\frac{d\phi\,(z)}{dz}\left[x + \frac{\partial\psi\,(\eta,\zeta)}{\partial y}\right] \tag{15}$$

where $(...)^{\mathrm{me}}$ and $(...)^{\mathrm{th}}$ denote mechanical and thermal parts, respectively.

Of particular interest is the axial mechanical strain $\varepsilon_{zz}^{\mathrm{me}}$, which consists of four terms. The first term in Eq. (12) accounts for the axial strain due to axial loading, the second term is due to the nonuniformity of the rate of twist $d\phi/dz$ (Vlasov [22]), the third term is due to pretwist, which introduces the dependence of the warping function $\psi(\eta,\zeta)$ on the axial coordinate z [see Eq. (3)], and the fourth term accounts for the axial thermal strain. It is this particular strain component that brings about the various interactions between axial- and twist-type of loading. Due to material isotropy, the temperature variation affects only the normal components of strain.

The corresponding stresses σ_{ij} are determined from the standard isotropic, linearly elastic constitutive equations that relate σ_{ij} to $\varepsilon_{ij}^{\mathrm{me}}$.

[1]The displacement components u, v in Reference [20] [given by Eq. (3) therein] have been mistyped.

Potential energy of the beam and the governing differential equations

The total elastic strain energy of the isotropic, linearly elastic beam is

$$U = \int_0^L \left\{ \iint \frac{E}{2} \left[\left(\varepsilon_{xx}^{\mathrm{me}}\right)^2 + \left(\varepsilon_{yy}^{\mathrm{me}}\right)^2 + \left(\varepsilon_{zz}^{\mathrm{me}}\right)^2 \right] \mathrm{dxdy} + \iint 2\,G \left[\left(\varepsilon_{xz}^{\mathrm{me}}\right)^2 + \left(\varepsilon_{yz}^{\mathrm{me}}\right)^2 \right] \mathrm{dxdy} \right\} \mathrm{dz} \quad (16)$$

where E is Young's modulus, G the elastic shear modulus, and the double integrals on (x, y) (or (η, ζ)) in all equations of the article are understood to be evaluated over the cross section.

Using the expressions (10)–(15) for the mechanical strain components and integrating by parts, we conclude after some lengthy but otherwise straightforward algebraic manipulations that the variation δU can be written in the form

$$\begin{aligned}
\delta U = & -\int_0^L \delta w_1 \left(E\,A \frac{d^2 w_1}{dz^2} + \alpha_0\,E\,S \frac{d^2\phi}{dz^2} - \alpha\,E\,A \frac{d\Delta\theta}{dz} \right) \mathrm{dz} \\
& - \int_0^L \delta\phi \left[\left(\alpha_0^2\,E\,K + G\,J\right) \frac{d^2\phi}{dz^2} - \alpha_0\,\alpha\,E\,S \frac{d\Delta\theta}{dz} + \alpha_0\,E\,S \frac{d^2 w_1}{dz^2} - E\,J_\omega \frac{d^4\phi}{dz^4} \right] \mathrm{dz} \\
& + \left[\delta w_1 \left(E\,A \frac{dw_1}{dz} + \alpha_0\,E\,S \frac{d\phi}{dz} - \alpha\,E\,A\,\Delta\theta \right) \right]_0^L \\
& + \left[\delta\phi \left(-E\,J_\omega \frac{d^3\phi}{dz^3} + \alpha_0\,E\,S \frac{dw_1}{dz} - \alpha\,\alpha_0\,E\,S\,\Delta\theta \right) \right]_0^L \\
& + \left[\frac{d\delta\phi}{dz} \left(E\,J_\omega \frac{d^2\phi}{dz^2} + \alpha_0\,E\,R \frac{d\phi}{dz} \right) \right]_0^L
\end{aligned} \quad (17)$$

where A is the cross-sectional area,

$$K = \frac{1}{\alpha_0^2} \iint \left(\frac{\partial\psi}{\partial z} \right)^2 \mathrm{d}\eta \mathrm{d}\zeta = \iint \left(\zeta \frac{\partial\psi}{\partial\eta} - \eta \frac{\partial\psi}{\partial\zeta} \right)^2 \mathrm{d}\eta \mathrm{d}\zeta \geq 0 \quad (18)$$

$$J = \iint \left[\left(\eta + \frac{\partial\psi}{\partial\zeta} \right)^2 + \left(-\zeta + \frac{\partial\psi}{\partial\eta} \right)^2 \right] \mathrm{d}\eta \mathrm{d}\zeta > 0 \quad (19)$$

$$J_\omega = \iint \psi^2\,(\eta, \zeta)\,\mathrm{d}\eta \mathrm{d}\zeta \geq 0 \quad (20)$$

$$R = \frac{1}{\alpha_0} \iint \psi \frac{\partial\psi}{\partial z} \mathrm{d}\eta \mathrm{d}\zeta = \iint \psi \left[\left(\frac{\partial\psi}{\partial\eta} \right)^2 + \left(\frac{\partial\psi}{\partial\zeta} \right)^2 \right] \mathrm{d}\eta \mathrm{d}\zeta \quad (21)$$

$$S = \frac{1}{\alpha_0} \iint \frac{\partial\psi}{\partial z} \mathrm{d}\eta \mathrm{d}\zeta = \iint \left[\left(\frac{\partial\psi}{\partial\eta} \right)^2 + \left(\frac{\partial\psi}{\partial\zeta} \right)^2 \right] \mathrm{d}\eta \mathrm{d}\zeta \geq 0 \quad (22)$$

and the warping function is normalized so that $\iint \psi\,(\eta, \zeta)\,\mathrm{d}\eta \mathrm{d}\zeta = 0$.

The quantities $(A, K, J, J_\omega, R, S)$ in Eqs. (18)–(22) are defined completely by the shape of the cross section. A detailed discussion of the aforementioned geometrical quantities and their physical interpretation is given by Kordolemis et al. [20]; a list of their values for various shapes of the cross section is presented in the Appendix.

Let p_z and m_z be the axially distributed force and torsional moment, respectively, applied along the beam, i.e.,

$$p_z = -\frac{dN\,(z)}{dz} \quad \text{and} \quad m_z = -\frac{dT\,(z)}{dz} \quad (23)$$

where $N(z)$ and $T(z)$ are the the axial force and torsional moment along the beam. The variation of the work δW of the external forces can be written in the form:

$$\delta W = \int_0^L \left(p_z\, \delta w_1 + m_z\, \delta\phi\right) \mathrm{d}z + [N\, \delta w_1]_0^L + [T\, \delta\phi]_0^L + \left[(-\bar{B})\, \delta \frac{d\phi}{dz}\right]_0^L \tag{24}$$

where $\bar{B}$ denotes the boundary values of the bimoment, which is defined as:

$$B(z) = -\int_A \sigma_{zz}(x, y, z)\psi(x, y, z)\, \mathrm{d}x\mathrm{d}y \text{ with } \sigma_{zz} = E\,\varepsilon_{zz} \text{ (\textit{Vlasov} [22]).}$$

The condition of minimum potential energy can be written in the form $\delta U - \delta W = 0$. Substituting Eqs. (17) and (24) in the condition $\delta U - \delta W = 0$, we arrive at the following Euler–Lagrange equations

$$\frac{d^2 w_1}{dz^2} + \frac{\alpha_0\, S}{A}\frac{d^2\phi}{dz^2} = -\frac{p_z}{E\,A} + \alpha\frac{d\Delta\theta}{dz} \tag{25}$$

$$-\ell^2\frac{d^4\phi}{dz^4} + \left(1 + \frac{\alpha_0^2\, K}{J}\frac{E}{G}\right)\frac{d^2\phi}{dz^2} + \frac{\alpha_0\, S}{J}\frac{E}{G}\frac{d^2 w_1}{dz^2} = -\frac{m_z}{G\,J} + \frac{\alpha_0\, S}{J}\frac{E}{G}\alpha\frac{d\Delta\theta}{dz} \tag{26}$$

and boundary conditions at the ends $z = 0$ and $z = L$:

$$w_1 = \bar{w}_1 \quad \text{or} \quad \frac{dw_1}{dz} + \frac{\alpha_0\, S}{A}\frac{d\phi}{dz} = \frac{\bar{N}}{E\,A} + \alpha\,\Delta\bar{\theta}, \quad \left(\text{with } \Delta\bar{\theta} = \Delta\theta(0) \text{ or } \Delta\theta(L)\right) \tag{27}$$

$$\phi = \bar{\phi} \quad \text{or} \quad -\ell^2\frac{d^3\phi}{dz^3} + \left(1 + \frac{\alpha_0^2\, K}{J}\frac{E}{G}\right)\frac{d\phi}{dz} + \frac{\alpha_0\, S}{J}\frac{E}{G}\frac{dw_1}{dz} = \frac{\bar{T}}{G\,J} + \frac{\alpha_0\, S}{J}\frac{E}{G}\alpha\,\Delta\bar{\theta} \tag{28}$$

$$\frac{d\phi}{dz} = \bar{\bar{\phi}} \quad \text{or} \quad \ell^2\frac{d^2\phi}{dz^2} + \frac{\alpha_0\, R}{J}\frac{E}{G}\frac{d\phi}{dz} = -\frac{\bar{B}}{G\,J} \tag{29}$$

where

$$\ell^2 = \frac{J_\omega}{J}\frac{E}{G} \tag{30}$$

and $(\bar{w}_1, \bar{\phi}, \bar{\bar{\phi}}, \bar{N}, \bar{T}, \bar{B})$ are the applied "loads" at the ends of the beam. Equations (25)–(29) define the boundary value problem that determines the unknown functions $w_1(z)$ and $\phi(z)$. It should be noted that the aforementioned boundary value problem can be derived from that listed by Kordolemis et al. [20], if dw_1/dz in [20] is replaced by $dw_1/dz - \alpha\,\Delta\theta$.

Guided by Eqs. (27), (28), and (29), we write

$$N(z) = E\,A\left[\frac{dw_1(z)}{dz} - \alpha\,\Delta\theta(z)\right] + \alpha_0\, S\,E\frac{d\phi(z)}{dz} \tag{31}$$

$$T(z) = G\,J\left[\frac{d\phi(z)}{dz} - \ell^2\frac{d^3\phi(z)}{dz^3}\right] + \alpha_0\, S\,E\left[\frac{dw_1(z)}{dz} - \alpha\,\Delta\theta(z)\right] + \alpha_0^2\, K\,E\frac{d\phi(z)}{dz} \tag{32}$$

$$B(z) = -G\,J\,\ell^2\frac{d^2\phi(z)}{dz^2} - \alpha_0\, R\,E\frac{d\phi(z)}{dz} \tag{33}$$

Then Eqs. (25) and (26) are equivalent to Eq. (23), i.e.,

$$\frac{dN(z)}{dz} = -p_z(z) \tag{34}$$

$$\frac{dT(z)}{dz} = -m_z(z) \tag{35}$$

and the boundary conditions (27), (28), and (29) take the form

$$w_1 = \bar{w}_1 \quad \text{or} \quad N = \bar{N} \tag{36}$$

$$\phi = \bar{\phi} \quad \text{or} \quad T = \bar{T} \tag{37}$$

$$\frac{d\phi}{dz} = \bar{\bar{\phi}} \quad \text{or} \quad B = \bar{B} \tag{38}$$

It is worth mentioning the tension–torsion coupling when the beam is pretwisted. If $\alpha_0 \neq 0$, the axial force $N(z)$ defined in Eq. (31) depends on the rotation $\phi(z)$. Similarly, the torsional moment $T(z)$ depends on the axial displacement $w_1(z)$, when $\alpha_0 \neq 0$. Also note that, for $\alpha_0 = 0$, the bimoment $B(z)$ is nonzero when the rate of twist $d\phi(z)/dz$ is not constant along the beam; however, for $\alpha_0 \neq 0$, a nonzero bimoment develops even for constant rate of twist $d\phi(z)/dz$ along the beam (see Eq. (33)). Equations (31)–(33) that relate the structural loads (N, T, B) to the generalized displacements (w_1, ϕ) can be viewed as structural constitutive equations for the beam.

The expression for the torsional moment $T(z)$ in Eq. (32) can be written as the sum of three terms:

$$T = T_{SV} + T_B + T_{\alpha_0} \tag{39}$$

$$T_{SV} = \int\int (x\,\sigma_{zy} - y\,\sigma_{zx})\,\mathrm{dxdy} = GJ\frac{d\phi(z)}{dz} \tag{40}$$

$$T_B = \frac{dB}{dz} = -GJ\,\ell^2\frac{d^3\phi(z)}{dz^3} - \alpha_0\,RE\frac{d^2\phi(z)}{dz^2} \tag{41}$$

$$T_{\alpha_0} = \int\int \sigma_{zz}\,\frac{\partial\psi}{\partial z}\,\mathrm{dxdy} = \alpha_0\,SE\left[\frac{dw_1(z)}{dz} - \alpha\,\Delta\theta(z)\right] + \alpha_0^2\,KE\frac{d\phi(z)}{dz} + \alpha_0\,RE\frac{d^2\phi(z)}{dz^2} \tag{42}$$

where $\sigma_{zz} = E\,\varepsilon_{zz}$, T_{SV} is the standard term that arises from the Saint-Venant shear stresses, T_B is the torque due to the nonuniformity of the rate of twist $d\phi/dz$ (Vlasov [22]), and T_{α_0} is the additional torque ("bishear") due to pretwist α_0 (see Simo and Vu-Quoc [23], Eq. (31c)). It should be noted that the shear stresses responsible for T_B and T_{α_0} appear to have no strain counterpart (see Eqs. (14) and (15)). The situation is analogous to that in the Bernoulli–Euler technical beam theory, where shear stresses are often calculated and the assumption of "plane sections" (shear strains are ignored) is used at the same time.

The problem in terms of $w_1(z)$

We use the approach of Kordolemis et al. [20] and formulate the problem in terms of $w_1(z)$ by eliminating $\phi(z)$ from Eqs. (25) and (26). Solving Eq. (25) for $d^2\phi/dz^2$ and substituting the result in Eq. (26), we arrive at the following fourth-order differential equation for $w_1(z)$:

$$g^2\frac{d^4w_1(z)}{dz^4} - \frac{d^2w_1(z)}{dz^2} = \frac{q(z)}{EA} + \alpha\left[g^2\frac{d^3\Delta\theta(z)}{dz^3} - \frac{d\Delta\theta(z)}{dz}\right] \tag{43}$$

where

$$q(z) = \frac{1}{c^2}\left(1 + \frac{\alpha_0^2\,K}{J}\frac{E}{G}\right)p_z(z) - g^2\frac{d^2p_z(z)}{dz^2} - \frac{\alpha_0\,ES}{c^2}\frac{m_z(z)}{GJ} \tag{44}$$

and

$$c^2 = 1 + \alpha_0^2\frac{E}{GJ}\left(K - \frac{S^2}{A}\right) \geq 0, \quad g^2 = \frac{\ell^2}{c^2} \tag{45}$$

To express the boundary conditions in terms of w_1 only, we use Eq. (25) to find

$$\frac{d^2\phi}{dz^2} = \frac{A}{\alpha_0\,S}\left(-\frac{p_z}{EA} - \frac{d^2w_1}{dz^2} + \alpha\frac{d\Delta\theta}{dz}\right) \quad \text{and} \quad \frac{d^3\phi}{dz^3} = \frac{A}{\alpha_0\,S}\left(-\frac{1}{EA}\frac{dp_z}{dz} - \frac{d^3w_1}{dz^3} + \alpha\frac{d^2\Delta\theta}{dz^2}\right) \tag{46}$$

Then, we combine Eqs. (27) and (28) first and Eqs. (27) and (29) next, to eliminate $d\phi/dz$ from them. Finally, we substitute Eq. (46) into the two equations resulting from the aforementioned eliminations and arrive at the following two boundary conditions at the ends $z = 0$ and $z = L$:

$$w_1 = \bar{w}_1 \quad \text{or} \quad -g^2 \frac{d^3 w_1}{dz^3} + \frac{dw_1}{dz} = \frac{\bar{P}}{EA} + \alpha \left(-g^2\, \Delta\bar{\theta}'' + \Delta\bar{\theta}\right)$$
$$\left(\text{with } \Delta\bar{\theta}'' = \left.\tfrac{d^2\Delta\theta}{dz^2}\right|_{z=0} \quad \text{or} \quad \left.\tfrac{d^2\Delta\theta}{dz^2}\right|_{z=L}\right) \tag{47}$$

and

$$\frac{dw_1}{dz} = \bar{\bar{w}}_1 \quad \text{or} \quad g^2 \frac{d^2 w_1}{dz^2} + h\frac{dw_1}{dz} = \frac{\bar{Y}}{EA} + \alpha \left(g^2\, \Delta\bar{\theta}' + h\, \Delta\bar{\theta}\right)$$
$$\left(\text{with } \Delta\bar{\theta}' = \left.\tfrac{d\Delta\theta}{dz}\right|_{z=0} \quad \text{or} \quad \left.\tfrac{d\Delta\theta}{dz}\right|_{z=L}\right) \tag{48}$$

where

$$\bar{P} = g^2\, \bar{p}_z' + \frac{1}{c^2}\left(1 + \frac{\alpha_0^2\, K}{J}\frac{E}{G}\right)\bar{N} - \frac{\alpha_0\, S}{c^2\, J}\frac{E}{G}\bar{T} \quad \left(\text{with } \bar{p}_z' = \left.\frac{dp_z}{dz}\right|_{z=0} \quad \text{or} \quad \left.\frac{dp_z}{dz}\right|_{z=L}\right) \tag{49}$$

$$\bar{Y} = -g^2\, \bar{p}_z + h\, \bar{N} + \frac{\alpha_0\, S}{c^2\, J}\frac{E}{G}\bar{B} \quad \left(\text{with } \bar{p}_z = p_z(0) \quad \text{or} \quad p_z(L)\right) \tag{50}$$

$$\bar{\bar{w}}_1 = \frac{\bar{N}}{EA} - \frac{\alpha_0\, S}{A}\bar{\bar{\phi}} + \alpha\, \Delta\bar{\theta} \tag{51}$$

$$h = \frac{\alpha_0\, R}{c^2\, J}\frac{E}{G} \tag{52}$$

The quantity h defined in Eq. (52) has dimensions of length and can be viewed as a *surface boundary material length parameter*. Kordolemis et al. [20] have pointed out that $|h| \le g$. The sign of h is the same as the sign of the pretwist α_0. In cross sections that have one axis of symmetry, the cross-sectional geometric parameter R vanishes and $h = 0$.

The quantities $\bar{P}$ and $\bar{Y}$ introduced in Eqs. (49) and (50) are "generalized end loads" and are defined in terms of the "traditional mechanical end loads" $(\bar{N}, \bar{T}, \bar{p}_z, \bar{B})$. We note that h enters the problem only when the generalized load $\bar{Y}$ is prescribed at one or both ends of the beam, i.e., when boundary condition (48b) is active; for example, in "fully constrained problems" where w_1 and dw_1/dz are prescribed at both ends, the axial displacement field $w_1(z)$ in the beam is independent of h.

We use the expressions in Eqs. (49) and (50) to define the generalized loads $P(z)$ and $Y(z)$ on any cross section along the beam:

$$P(z) = g^2 \frac{dp_z(z)}{dz} + \frac{1}{c^2}\left(1 + \frac{\alpha_0^2\, K}{J}\frac{E}{G}\right) N(z) - \frac{\alpha_0\, S}{c^2\, J}\frac{E}{G} T(z) \tag{53}$$

$$Y(z) = -g^2\, p_z(z) + h\, N(z) + \frac{\alpha_0\, S}{c^2\, J}\frac{E}{G} B(z) \tag{54}$$

We can also use Eqs. (31)–(33) and (46) to arrive at the following alternative relations for the generalized loads $P(z)$ and $Y(z)$:

$$P(z) = EA\left(1 - g^2 \frac{d^2}{dz^2}\right)\left[\frac{dw_1(z)}{dz} - \alpha\, \Delta\theta(z)\right] \tag{55}$$

$$Y(z) = EA\left(h + g^2 \frac{d}{dz}\right)\left[\frac{dw_1(z)}{dz} - \alpha\, \Delta\theta(z)\right] \tag{56}$$

Equations (43), (47), and (48) define the boundary value problem that determines $w_1(z)$. Once $w_1(z)$ has been determined, the solution is completed with the calculation of $\phi(z)$ from the differential equation (25):

$$\frac{d^2\phi(z)}{dz^2} = -\frac{A}{\alpha_0\, S}\left\{\frac{d}{dz}\left[\frac{dw_1(z)}{dz} - \alpha\,\Delta\theta(z)\right] + \frac{p_z(z)}{EA}\right\} \tag{57}$$

and the appropriate boundary condition at $z = 0$ and L.

The general solution of Eqs. (43) and (57) is

$$\begin{aligned} w_1(z) &= c_1\, L\, \sinh\frac{z}{g} + c_2\, L\, \cosh\frac{z}{g} + c_3\, z + c_4\, L + \alpha\int\Delta\theta(z)\mathrm{d}z \\ &\quad - \frac{g}{EA}\int_0^z\left(\frac{z-\zeta}{g} - \sinh\frac{z-\zeta}{g}\right) q(\zeta)\,\mathrm{d}\zeta \end{aligned} \tag{58}$$

and

$$\phi(z) = -\frac{A}{\alpha_0\, S}\left\{w_1(z) - \alpha\int\Delta\theta(z)\,\mathrm{d}z + \frac{1}{EA}\int\left[\int p_z(z)\,\mathrm{d}z\right]\mathrm{d}z\right\} + c_5\frac{z}{L} + c_6 \tag{59}$$

where $(c_1, c_2, c_3, c_4, c_5, c_6)$ are dimensionless constants to be determined from the boundary conditions.

In "Analogy with one-dimensional gradient thermoelasticity" section, the one-dimensional gradient linear thermoelastic model of a homogeneous and isotropic bar under axial loading is presented and the direct analogy with the approach of this section is highlighted. In "Cases studies for various boundary conditions" section, analytical expressions of the displacement field $w_1(z)$ and the generalized loads (P, Y) are developed for various boundary conditions and a linear thermal load. It is also demonstrated that the interplay between the axial force N, torque T, and the bimoment B initiates the actuating character of the beam through thermal loading.

Analogy with one-dimensional gradient thermoelasticity

We consider quasi-static strain-gradient thermoelasticity and let τ_{ij} be the components of the Cauchy stress tensor, μ_{ijk} the components of the double-stress tensor, and F_i the components of the body force (force per unit volume). In a Cartesian coordinate system $(x_1, x_2, x_3) = (x, y, z)$, the equations of equilibrium take the form (Filopoulos et al. [24], [25]),

$$\left(\tau_{ji} - \mu_{kji,k}\right)_{,j} + F_i = 0 \tag{60}$$

The associated kinematic and traction boundary conditions are

$$u_i = \bar{u}_i \quad \text{or} \quad \bar{P}_i = n_j\left(\tau_{ji} - \mu_{kij,k}\right) - D_j\left(n_k\,\mu_{kji}\right) + \left(D_p n_p\right) n_j\, n_k\, \mu_{kji} \tag{61}$$

$$u_{i,j}\, n_j = \bar{\bar{u}}_i \quad \text{or} \quad \bar{Y}_i = n_k\, n_j\, \mu_{jki} \tag{62}$$

where $\bar{\mathbf{u}}$ and $\bar{\bar{\mathbf{u}}}$ are prescribed displacement and normal derivatives of displacements, $\bar{\mathbf{P}}$ prescribed generalized tractions, $\bar{\mathbf{Y}}$ prescribed generalized double tractions, $\mathbf{n}$ the unit outward normal to the boundary, $D_j = \left(\delta_{jk} - n_j\, n_k\right)\frac{\partial}{\partial x_k}$, δ_{jk} the Kronecker delta, repeated indices imply summation, and a comma followed by a subscript, say i, denotes partial differentiation with respect to the spatial coordinate x_i, e.g., $A_{,i} = \partial A/\partial x_i$.

We consider the one-dimensional strain gradient linear thermoelastic problem of a homogeneous and isotropic bar in tension/compression (see also Kordolemis et al. [20]). Poisson's ratio is assumed to vanish and the only nonzero displacement component is $u_z(z)$, where z is the direction of the bar axis. Then, the only nonzero components of strain, stress, and double stress are $\varepsilon_{zz} = du_z/dz$, τ_{zz}, and μ_{zzz}. The equilibrium equation (60) and the boundary conditions (61) and (62) at the ends of the bar are written as:

$$\frac{d\tau_{zz}}{dz} - \frac{d^2\mu_{zzz}}{dz^2} + F_z = 0 \tag{63}$$

and at the ends $z = 0$ and $z = L$

$$u_z = \bar{u}_z \quad \text{or} \quad \bar{P}_z = \tau_{zz} - \frac{d\mu_{zzz}}{dz} \tag{64}$$

$$\frac{du_z}{dz} = \bar{\bar{u}}_z \quad \text{or} \quad \bar{Y}_z = \mu_{zzz} \tag{65}$$

We use the results of Filopoulos et al. [24],[25] for the special case of one-dimensional linear strain gradient thermoelasticity, as it was done in the absence of thermal loads by Tsepoura et al. [26], and write the thermoelastic constitutive equations in the form

$$\tau_{zz} = E\left(\varepsilon_{zz}^{\text{me}} + h\frac{d\varepsilon_{zz}^{\text{me}}}{dz}\right), \quad \mu_{zzz} = E\left(h\,\varepsilon_{zz}^{\text{me}} + g^2\frac{d\varepsilon_{zz}^{\text{me}}}{dz}\right), \quad \text{with} \quad \varepsilon_{zz}^{\text{me}} = \frac{du_z}{dz} - \alpha\,\Delta\theta \tag{66}$$

where E is Young's modulus, (h, g) "material lengths," α the coefficient of thermal expansion, and $\Delta\theta$ the change of temperature along the bar. The simplified constitutive equations (66) have not been explicitly presented before, but they are tacitly included by Filopoulos et al. [24, 25]. The constitutive equations (66) can be written in the form:

$$\tau_{zz}(z) = E\left(\frac{du_z}{dz} + h\frac{d^2u_z}{dz^2}\right) - \alpha\,E\left(\Delta\theta + h\frac{d\Delta\theta}{dz}\right) \tag{67}$$

$$\mu_{zzz}(z) = E\left(h\frac{du_z}{dz} + g^2\frac{d^2u_z}{dz^2}\right) - \alpha\,E\left(h\,\Delta\theta + g^2\frac{d\Delta\theta}{dz}\right) \tag{68}$$

Substituting the above expressions for τ_{zz} and μ_{zzz} in the governing equilibrium equation and the boundary conditions (63)–(65), we arrive at the following boundary value problem for $u_z(z)$:

$$g^2\frac{d^4u_z(z)}{dz^4} - \frac{d^2u_z(z)}{dz^2} = \frac{f_z(z)}{EA} + \alpha\left[g^2\frac{d^3\Delta\theta(z)}{dz^3} - \frac{d\Delta\theta(z)}{dz}\right] \tag{69}$$

and at the ends $z = 0$ and $z = L$:

$$u_z = \bar{u}_z \quad \text{or} \quad -g^2\frac{d^3u_z}{dz^3} + \frac{du_z}{dz} = \frac{\bar{P}_z}{EA} + \alpha\left(-g^2\frac{d^2\Delta\bar{\theta}}{dz^2} + \Delta\bar{\theta}\right) \tag{70}$$

$$\frac{du_z}{dz} = \bar{\bar{u}}_z \quad \text{or} \quad g^2\frac{d^2u_z}{dz^2} + h\frac{du_z}{dz} = \frac{\bar{Y}_z}{EA} + \alpha\left(g^2\frac{d\Delta\bar{\theta}}{dz} + h\,\Delta\bar{\theta}\right) \tag{71}$$

where A is the cross-sectional area and $f_z(z) = F_z(z)\,A$ the axial body force per unit length of the bar.

Kordolemis et al. [20] have shown that the conditions $g \geq 0$ and $|h| \leq g$ are required for uniqueness of solution.

Equations (69)–(71) compare directly to Eqs. (43), (47), and (48) and establish a direct analogy between the thermal problem of the pretwisted beam and the one-dimensional strain gradient thermoelastic continuum, provided the substitutions listed in Table 1 are made.

Table 1. Equivalent quantities between the pretwisted beam and the one-dimensional gradient thermoelastic continuum.

Variable	Pretwisted beam	Gradient thermoelasticity
Axial displacement	$w_1\,(z)$	$u_z\,(z)$
Volume material length	$\sqrt{\frac{1}{c^2}\frac{J_\omega}{J}\frac{E}{G}}$	g
Surface material length	$\frac{\alpha_0}{c^2}\frac{R}{J}\frac{E}{G}$	h
Body force	$q(z) = \frac{1}{c^2}\left(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G}\right)p_z(z) - g^2\frac{d^2p_z(z)}{dz^2} - \frac{\alpha_0\,E\,S}{c^2}\frac{m_z(z)}{GJ}$	$f_z(z) = F_z(z)\,A$
Generalized traction	$\bar{P} = g^2\,\bar{p}_z' + \frac{1}{c^2}\left(1 + \frac{\alpha_0^2 S}{J}\frac{E}{G}\right)\bar{N} - \frac{\alpha_0\,S}{c^2\,J}\frac{E}{G}\bar{T}$	$\bar{P}_z$
Generalized double traction	$\bar{Y} = -g^2\,\bar{p}_z + h\,\bar{N} + \frac{\alpha_0\,S}{c^2\,J}\frac{E}{G}\bar{B}$	$\bar{Y}_z$
	$c^2 = 1 + \alpha_0^2\frac{E}{GJ}\left(K - \frac{S^2}{A}\right), \quad g \geq 0, \quad -g \leq h \leq g$	

Case studies for various boundary conditions

In this section, we present various case studies for the problem of a pretwisted beam subjected to thermal loads under different boundary conditions. We want to examine the effects of temperature variation $\Delta\theta(z)$ on the mechanical behavior of the pretwisted beam. We assume that the mechanical loads (N, T) take constant values along the pretwisted beam, i.e., $N(z) = N =$ const. and $T(z) = T =$ const. $(p_z = 0, m_z = 0)$. The corresponding values of the generalized loads (P, Y) are

$$P = \frac{1}{c^2}\left(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G}\right) N - \frac{\alpha_0 S}{c^2 J}\frac{E}{G} T = \text{const.} \tag{72}$$

$$Y(z) = h N + \frac{\alpha_0 S}{c^2 J}\frac{E}{G} B(z) \tag{73}$$

We note that P is constant along the beam, whereas Y may vary with z due to the bimoment $B(z)$. The bimoment takes nonzero values when there is pretwist or the rate of twist $d\phi/dz$ is not constant along the beam (Vlasov [22]) and is determined from Eq. (33):

$$B(z) = -\ell^2 G J \frac{d^2\phi(z)}{dz^2} - \alpha_0 R E \frac{d\phi(z)}{dz}.$$

To examine the effects of temperature variation $\Delta\theta(z)$ on the mechanical behavior of the pretwisted beam, we consider a linear temperature variation along the beam of the form

$$\Delta\theta(z) = \Delta\theta_0 \left(1 + D\frac{z}{L}\right) \tag{74}$$

where $\Delta\theta_0$ is the magnitude of temperature change at $z = 0$ and D is a dimensionless constant that controls the temperature gradient along the beam, i.e., $\Delta\theta(0) = \Delta\theta_0$ and $\Delta\theta(L) = \Delta\theta_0(1 + D)$. Homogeneous mechanical boundary conditions are used in all problems analyzed, i.e., we assume that $(w_1 = 0$ or $P = 0)$ and $(dw_1/dz = 0$ or $Y = 0)$ at the ends of the beam, so that the only applied "load" on the beam is the aforementioned temperature field $\Delta\theta(z)$. In view of the linearity of the problem and the absence of any mechanical loads, the solution is proportional to $\alpha\,\Delta\theta_0$.

It is emphasized that vanishing of the generalized loads (P, Y) at the ends of the beam does **not** necessarily mean that the corresponding end values of the true mechanical loads (N, T) vanish as well. In fact, the corresponding values of N and T at the ends of the beam are determined from Eqs. (72) and (73). In the first three examples, the beam is constrained axially at both ends $(w_1(0) = w_1(L) = 0)$ and the corresponding reactions are studied. In the last two examples, the end at $z = 0$ is fully constrained $\left(w_1(0) = \left.\frac{dw_1}{dz}\right|_{z=0} = 0\right)$ and the response at the other end is studied.

The solution is determined by solving the boundary value problem defined by Eqs. (43), (47), and (48). We recall that the general solution for $w_1(z)$ and $\phi(z)$ is of the form (see Eqs. (58) and (59))

$$w_1(z) = c_1 L \sinh\frac{z}{g} + c_2 L \cosh\frac{z}{g} + c_3 z + c_4 L + \alpha\,\Delta\theta_0\, z\left(1 + D\frac{z}{2L}\right) \tag{75}$$

and

$$\phi(z) = -\frac{A L}{\alpha_0 S}\left(c_1 \sinh\frac{z}{g} + c_2 \cosh\frac{z}{g} + c_3\frac{z}{L} + c_4\right) + c_5 z + c_6 \tag{76}$$

The corresponding constant axial force N is determined from Eq. (31):

$$N = \alpha_0\, c_5\, E S \tag{77}$$

The results show that the nonclassical boundary conditions that enter the problem of the pretwisted beam may induce an interesting drilling type of actuation that has been observed in various biosystems and will be discussed after the examples.

Case 1: Beam with fully constrained ends

We consider a beam with both ends fully constrained as shown in Figure 2 and study the generalized forces $P(z)$ and $Y(z)$ that develop along the beam.

The boundary conditions in this case read

$$w_1(0) = w_1(L) = 0 \quad \text{and} \quad \left.\frac{dw_1}{dz}\right|_{z=0} = \left.\frac{dw_1}{dz}\right|_{z=L} = 0 \tag{78}$$

Using these boundary conditions in the general solution (75), we find

$$w_1(z) = -\alpha\,\Delta\theta_0\,\frac{L\,D}{2}\left\{\frac{z}{L}\left(1-\frac{z}{L}\right) - \frac{g}{L}\left[\sinh\frac{z}{g} - 2\coth\frac{L}{2g}\left(\sinh\frac{z}{2g}\right)^2\right]\right\} \tag{79}$$

In this case, the ends of the beam are fully constrained, there are no boundary conditions in terms of Y, and the solution for $w_1(z)$ is independent of the surface material length h [see discussion in paragraph before Eq. (33) in "Problem formulation" section]. If the temperature along the length of the beam is constant, i.e., if $D = 0$, Eq. (79) implies that $w_1(z) = 0$, in agreement with classical thermoelasticity. Also in the limit $g \to 0$, we recover the classical solution of linear thermoelasticity:

$$\lim_{g\to 0} w_1(z) = -\alpha\,\Delta\theta_0\,\frac{D}{2} z\left(1-\frac{z}{L}\right) \tag{80}$$

Figure 3 shows the variation of $w_1(z)$ for different values of the ratio g/L. Note that, as the internal length g increases, the axial displacement w_1 decreases, thus indicating a stiffening effect when $g \neq 0$.

The generalized axial load $P(z)$ and the generalized double force $Y(z)$ can be determined from Eqs. (55) and (56). The resulting value for the generalized load $P(z)$ is constant as expected:

$$P = -\alpha\,\Delta\theta_0\,E\,A\left(1+\frac{D}{2}\right) \tag{81}$$

The sign of P is the opposite of the sign of $\Delta\theta_0\left(1+\frac{D}{2}\right)$.

The generalized force $Y(z)$ in this case reads

$$Y(z) = -\alpha\,\Delta\theta_0\,E\,A\,L\left[\frac{h}{L}\left(1+\frac{D}{2}\right) + \frac{D}{2\sinh\frac{L}{2g}}\left(\frac{g}{L}\cosh\frac{L-2z}{2g} - \frac{h}{L}\sinh\frac{L-2z}{2g}\right)\right] \tag{82}$$

We also have that

$$\lim_{g\to 0} Y(z) = -\alpha\,\Delta\theta_0\,E\,A\,h\left(1+\frac{D}{2}\right) \quad \text{and} \quad Y(z)|_{h=0} = -\alpha\,\Delta\theta_0\,E\,A\,\frac{g\,D}{2\sinh\frac{L}{2g}}\cosh\frac{L-2z}{2g} \tag{83}$$

When both material length vanish ($g = h = 0$), the generalized double force vanishes ($Y = 0$), i.e., we recover the classical thermoelastic solution, which does not involve double forces. When $h = 0$, the generalized double force $Y(z)$ is defined by (83b) and its sign is the opposite of the sign of $\Delta\theta_0\,D$.

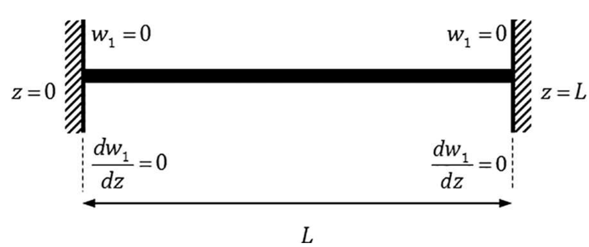


Figure 2. A beam with clamped ends (Case 1).

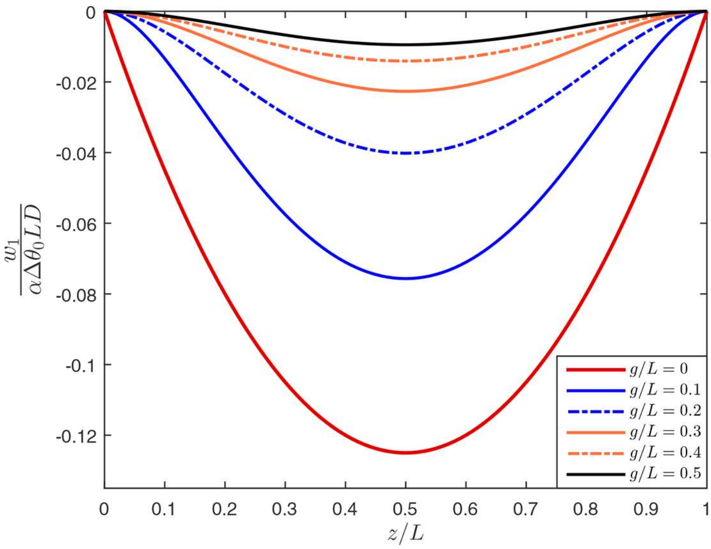


Figure 3. The axial displacement w_1 along the beam for various values of the ratio $\frac{g}{L}$ (Case 1).

Figure 4 shows the dependence of $Y(z)$ along the beam with the length scale g. As expected, the magnitude of the generalized double force increases when the material length g increases.

Figure 5 shows the dependence of the generalized double force $Y(z)$ on the surface material length h. According to Figure 5, it is possible for the generalized double force Y to have different signs at the two ends of the beam; also the sign of h, which is the same as the sign of the pretwist α_0, can influence the sign of Y at the ends of the beam. In the present example, the ends of the beam are fully constrained ($w_1 = 0$ and $dw_1/dz = 0$ at both ends), and the beam does not have any kinematical freedom at its

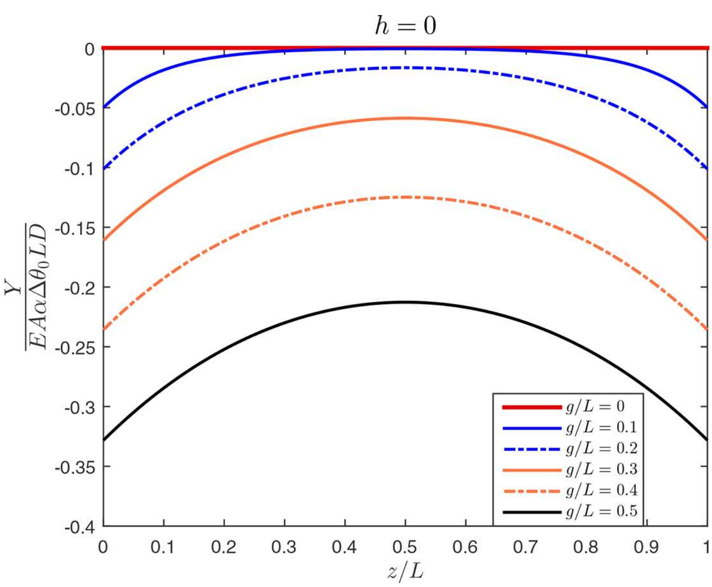


Figure 4. The distribution of the generalized axial double force Y for various values of g for $h = 0$ (Case 1).

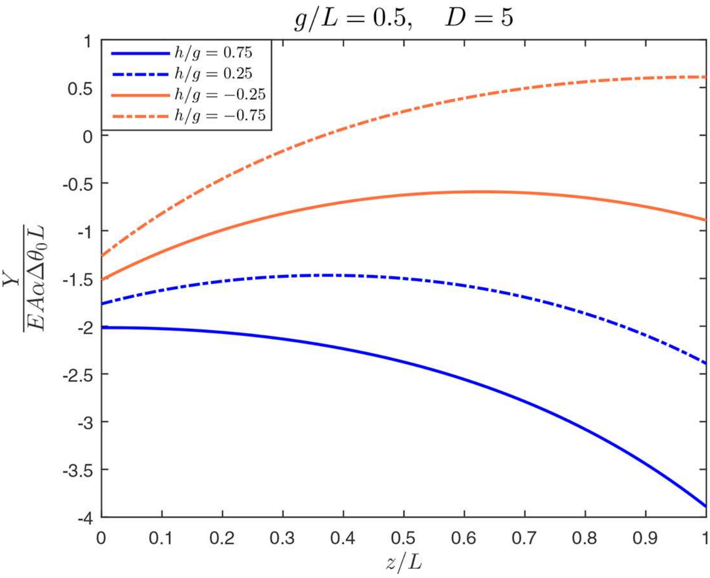


Figure 5. Variation of the generalized double force $Y(z)$ on h for $g/L = 0.5$ and $D = 5$ (Case 1).

ends; yet, the results of Figure 5 indicate that by controlling the microstructural parameters h and g, which can be achieved by changing the geometrical parameters of the cross section and the amount and sign of pretwist, we can control the magnitude and sign of Y, thus paving the way for the actuating capabilities of the beam. Such possibilities are explored in the following examples, in which different boundary conditions are used.

Case 2: Beam with fixed ends and $Y = 0$ at both ends

In this second example, the constraints at both ends are relaxed a bit and the conditions $\left.\frac{dw_1}{dz}\right|_{z=0} = \left.\frac{dw_1}{dz}\right|_{z=L} = 0$ of Case 1 are replaced by $Y(0) = Y(L) = 0$. The boundary conditions now are (Figure 6)

$$w_1(0) = w_1(L) = 0 \quad \text{and} \quad Y(0) = Y(L) = 0 \tag{84}$$

and the constants in Eq. (75) take the values

$$c_1 = -\frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L},\frac{h}{L}\right)}\left(1+\frac{D}{2}\right)\frac{g\,h}{L^2}\left[\frac{g}{L}\left(\cosh\frac{L}{g}-1\right)+\frac{h}{L}\sinh\frac{L}{g}\right] \tag{85}$$

$$c_2 = -c_4 = \frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L},\frac{h}{L}\right)}\left(1+\frac{D}{2}\right)\frac{g\,h}{L^2}\left[\frac{h}{L}\left(\cosh\frac{L}{g}-1\right)+\frac{g}{L}\sinh\frac{L}{g}\right] \tag{86}$$

$$c_3 = -\frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L},\frac{h}{L}\right)}\left(1+\frac{D}{2}\right)\frac{g^2-h^2}{L^2}\sinh\frac{L}{g} \tag{87}$$

where

$$\Delta\left(\frac{g}{L},\frac{h}{L}\right) = 2\,\frac{g\,h^2}{L^3}\left(\cosh\frac{L}{g}-1\right)+\frac{g^2-h^2}{L^2}\sinh\frac{L}{g} \geq 0 \tag{88}$$

In the limit $g \to 0$, the classical thermoelastic solution (80) for $w_1(z)$ is recovered.

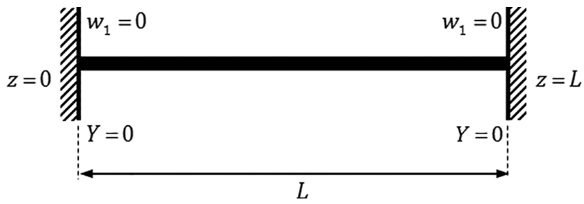


Figure 6. Beam with fixed ends and $Y = 0$ at both ends (Case 2).

As in Case 1, when the material length $g \neq 0$, the displacement field can change substantially in magnitude and sign, depending on the particular values of $\Delta\theta_0$, D, g, and h. Also, a larger value of g results in stiffer response (smaller w_1).

The generalized axial force P in this case takes the value

$$P = -\alpha\,\Delta\theta_0 \frac{EA}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}(2+D)\frac{g^2-h^2}{L^2}\sinh\frac{L}{g} \tag{89}$$

We note that $P(-h) = P(h)$ and that the sign of P is the opposite of the sign of $\Delta\theta_0(2+D)$. Also in the limit $g \to 0$, we recover the value of P given in Eq. (81).

The generalized double force Y in this case takes the form

$$Y(z) = -\alpha\,\Delta\theta_0 \frac{4EAL}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}(2+D)\frac{h(g^2-h^2)}{L^3}\sinh\frac{L}{2g}\sinh\frac{L-z}{2g}\sinh\frac{z}{2g} \tag{90}$$

We note that $Y(-h) = -Y(h)$, so that $Y|_{h=0} = 0$, and that the sign of Y is the opposite of the sign of $\Delta\theta_0(2+D)h$. Also, in the limit $g \to 0$, Y takes again the value given in Eq. (83a).

It is also interesting to note that in this case, when $h = \pm g$, both P and $Y(z)$ vanish.

Case 3: Beam clamped at both ends with $\left.\frac{dw_1}{dz}\right|_{z=0} = 0$ and $Y(L) = 0$

In this problem, we use "nonsymmetric" boundary conditions at the ends (Figure 7):

$$w_1(0) = w_1(L) = 0 \quad \text{and} \quad \left.\frac{dw_1}{dz}\right|_{z=0} = 0, \quad Y(L) = 0 \tag{91}$$

and the constants in Eq. (75) take the values

$$c_1 = \frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}\frac{g}{L}\left[-2\frac{gh}{L^2} + \frac{g}{L}\left(2\frac{h}{L} + D\right)\cosh\frac{L}{g} + D\frac{h}{L}\sinh\frac{L}{g}\right] \tag{92}$$

$$c_2 = -c_4 = -\frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}\frac{g}{L}\left[-(2+D)\frac{h}{L} + D\frac{h}{L}\cosh\frac{L}{g} + \frac{g}{L}\left(2\frac{h}{L} + D\right)\sinh\frac{L}{g}\right] \tag{93}$$

$$c_3 = \frac{\alpha\,\Delta\theta_0}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}\left\{2\frac{gh}{L^2} + \frac{g}{L}\left(-2\frac{h}{L} + 2 + D\right)\cosh\frac{L}{g} + \left[-2\frac{g^2}{L^2} + (2+D)\frac{h}{L}\right]\sinh\frac{L}{g}\right\} \tag{94}$$

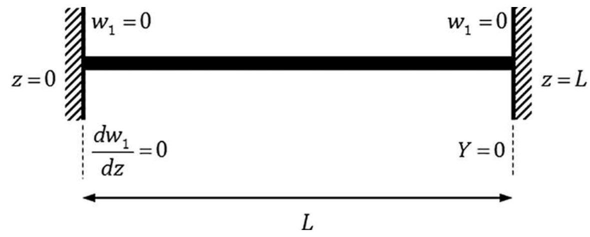


Figure 7. Beam clamped at both ends with $\left.\frac{dw_1}{dz}\right|_{z=0} = 0$ and $Y(L) = 0$ (Case 3).

where

$$\Delta\left(\frac{g}{L}, \frac{h}{L}\right) = 2\left[2\frac{g\,h}{L^2} + \frac{g}{L}\left(1 - 2\frac{h}{L}\right)\cosh\frac{L}{g} + \left(\frac{h}{L} - \frac{g^2}{L^2}\right)\sinh\frac{L}{g}\right] \tag{95}$$

In the limit $g \to 0$, the classical thermoelastic solution (80) for $w_1(z)$ is recovered.

The generalized axial force P in this case takes the value

$$P = -\alpha\,\Delta\theta_0\,\frac{E\,A}{\Delta\left(\frac{g}{L}, \frac{h}{L}\right)}\left\{2\frac{g\,h}{L^2} + \frac{g}{L}\left(-2\frac{h}{L} + 2 + D\right)\cosh\frac{L}{g} + \left[-2\frac{g^2}{L^2} + (2 + D)\frac{h}{L}\right]\sinh\frac{L}{g}\right\} \tag{96}$$

In the limit $g \to 0$, we recover the value of P given in Eq. (81) and

$$P|_{h=0} = -\alpha\,\Delta\theta_0\,\frac{E\,A}{2}\,\frac{(2 + D)\cosh\frac{L}{g} - 2\frac{g}{L}\sinh\frac{L}{g}}{\cosh\frac{L}{g} - \frac{g}{L}\sinh\frac{L}{g}}$$

Figure 8 shows the variation of P with the ratio h/g for different values of D and for $g/L = 0.5$. It appears that the effects of h on P become important only for large values of the temperature gradient (measured

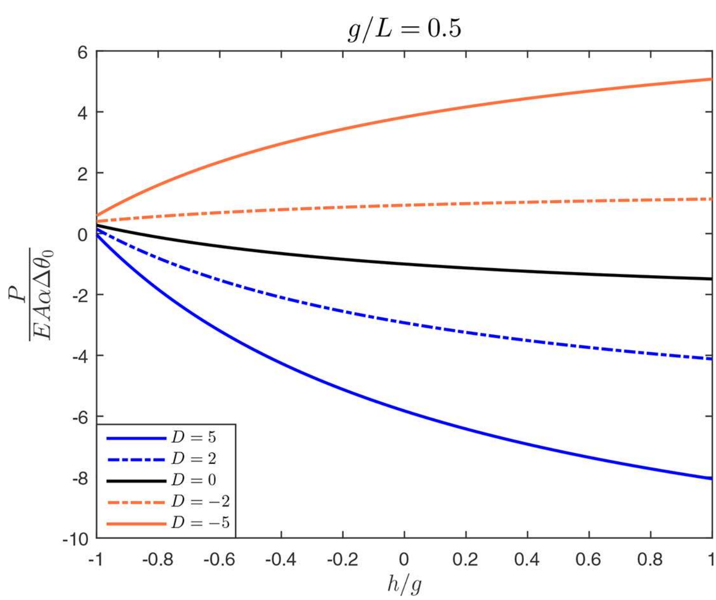


Figure 8. Variation of the generalized axial force P with the the ratio h/g and for different values D and for $g/L = 0.5$ (Case 3).

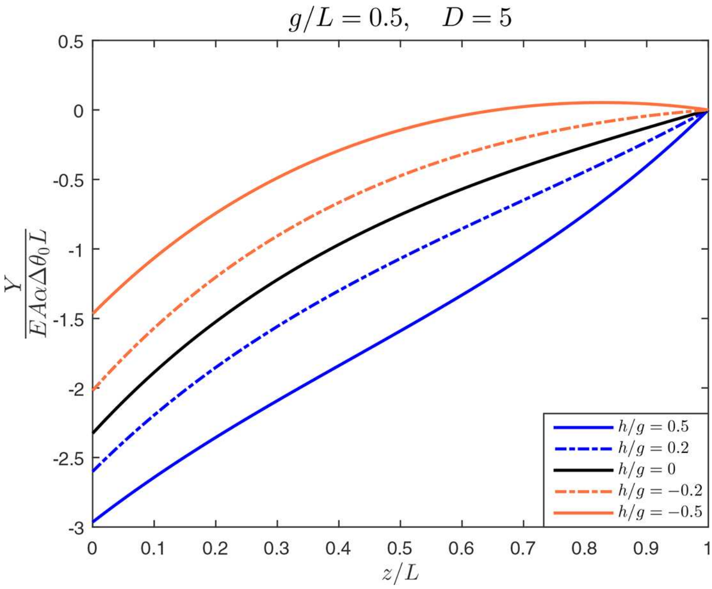


Figure 9. The distribution of the generalized axial double force Y for different values of the ratio h/g and for $g/L = 0.5, D = 5$ (Case 3).

by D), along the beam. The generalized double force Y in this case takes the form

$$Y(z) = -\alpha\,\Delta\theta_0\,\frac{2\,E\,A\,L}{\Delta\left(\frac{g}{L},\frac{h}{L}\right)}\sinh\frac{L-z}{2g}\left\{\left(2\frac{g^2 h}{L^3} + D\frac{g^2-h^2}{L^2}\right)\cosh\frac{L-z}{2g} + \right.$$
$$+\frac{h}{L}\left[\left(-2\frac{g^2}{L^2} + (2+D)\frac{h}{L}\right)\cosh\frac{L+z}{2g} - 2\frac{g\,h}{L^2}\sinh\frac{L-z}{2g}\right.$$
$$\left.\left. + \frac{g}{L}\left(-2\frac{h}{L} + 2 + D\right)\sinh\frac{L+z}{2g}\right]\right\} \tag{97}$$

In the limit $g \to 0$, Y takes again the value given in Eq. (83a) and

$$Y(z)|_{h=0} = -\alpha\,\Delta\theta_0\,\frac{E\,A\,D\,g}{2}\,\frac{\sinh\frac{L-z}{g}}{\cosh\frac{L}{g} - \frac{g}{L}\sinh\frac{L}{g}} \tag{98}$$

Figure 9 shows the variation of the double force $Y(z)$ along the beam for different values of the ratio h/g and for $g/L = 0.5$ and $D = 5$. It appears that the surface material length h can influence substantially the generalized axial double force Y along the beam.

Case 4: Beam fully constrained at one end and free at the other

We consider a beam fully constrained at $z = 0$ and load free at $z = L$ (Figure 10). We recall that the prescribed temperature field $\Delta\theta(z)$ is the only driving force.

The boundary conditions for this case read

$$w_1(0) = 0, \qquad \left.\frac{dw_1}{dz}\right|_{z=0} = 0 \quad \text{and} \quad P(L) = 0, \quad Y(L) = 0 \tag{99}$$

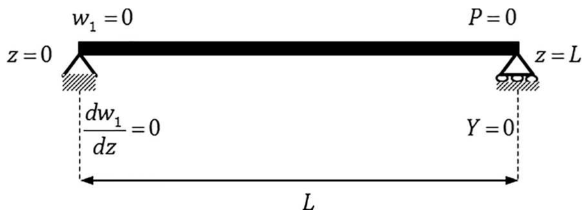


Figure 10. Beam fully constrained at one end and free at the other (Case 4).

and the axial displacement $w_1(z)$ takes the form

$$w_1(z) = \alpha\,\Delta\theta_0\,L\left[\frac{z}{L}\left(1 + D\frac{z}{2\,L}\right) - \frac{g}{L}\sinh\frac{z}{g} + \frac{g}{L}\frac{h\,\cosh\frac{L}{g} + g\sinh\frac{L}{g}}{g\cosh\frac{L}{g} + h\,\sinh\frac{L}{g}}\left(\cosh\frac{z}{g} - 1\right)\right] \tag{100}$$

In the limit $g \to 0$, we recover the corresponding solution of linear thermoelasticity:

$$\lim_{g\to 0} w_1(z) = \alpha\,\Delta\theta_0\,z\left(1 + D\frac{z}{2\,L}\right) \tag{101}$$

Careful examination of Eq. (100) reveals that larger values of g result in lower axial displacements (stiffening effect).

The generalized axial force P takes a constant value and in view of the boundary condition (99c) vanishes along the beam. The generalized double force $Y(z)$ takes the value

$$Y(z) = \alpha\,\Delta\theta_0\,E\,A\,\frac{g^2 - h^2}{g\,\cosh\frac{L}{g} + h\,\sinh\frac{L}{g}}\,\sinh\frac{L-z}{g} \tag{102}$$

Note that Y is independent of the temperature gradient D in this case. Also

$$\lim_{g\to 0} Y(z) = 0 \quad \text{and} \quad Y(z)|_{h=0} = \alpha\,\Delta\theta_0\,\frac{E\,A\,g}{\cosh\frac{L}{g}}\,\sinh\frac{L-z}{g} \tag{103}$$

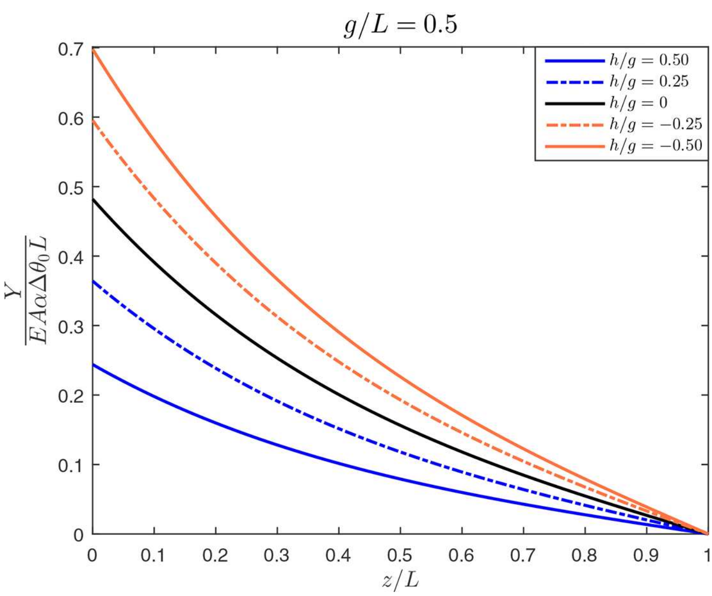


Figure 11. The distribution of the generalized axial double force Y for different values of the ratio h/g and for $g/L = 0.5$ (Case 4).

Figure 11 shows the variation of the double force $Y(z)$ along the beam for different values of the ratio h/g and for $g/L = 0.5$. It appears that the sign of h, i.e., the sign of the pretwist α_0, affects strongly the magnitude of $Y(z)$.

Case 5: Beam joint at one end and with a roller at the other with $\left.\frac{dw_1}{dz}\right|_{z=0,L}$

The boundary conditions is this case are (Figure 12):

$$w_1(0) = 0, \quad P(L) = 0 \quad \text{and} \quad \left.\frac{dw_1}{dz}\right|_{z=0} = \left.\frac{dw_1}{dz}\right|_{z=L} = 0 \tag{104}$$

The axial displacement in this case has the form

$$w_1(z) = \alpha\,\Delta\theta_0\,L\left\{\frac{z}{L}\left(1 + D\frac{z}{2L}\right) - \frac{\frac{g}{L}}{\sinh\frac{L}{g}}\left[\cosh\frac{L}{g} - \cosh\frac{L-z}{g} + (1+D)\left(\cosh\frac{z}{g} - 1\right)\right]\right\} \tag{105}$$

The axial displacement field $w_1(z)$ is independent of h, because the boundary conditions do not involve the generalized double force Y. In the limit $g \to 0$, we recover the corresponding solution of linear thermoelasticity (Eq. (101)).

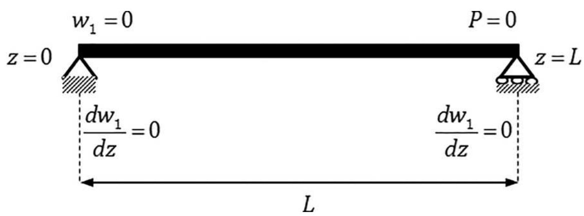


Figure 12. Beam with a joint at one end and a roller to the other with $\left.\frac{dw_1}{dz}\right|_{z=0,L} = 0$ (Case 5).

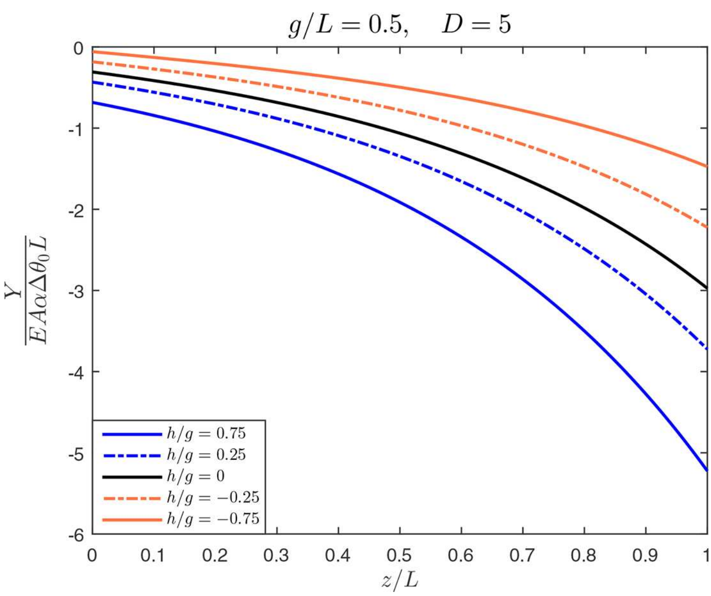


Figure 13. The distribution of the generalized axial double force Y for different values of the ratio h/g and for $g/L = 0.5$ (Case 5).

The generalized axial force P vanishes and $Y(z)$ is

$$Y(z) = -\alpha\,\Delta\theta_0 \frac{E\,A\,L}{\sinh\frac{L}{g}} \left\{ -\frac{g}{L}\cosh\frac{L-z}{g} + (1+D)\frac{g}{L}\cosh\frac{z}{g} + \frac{h}{L}\left[\sinh\frac{L-z}{g} + (1+D)\sinh\frac{z}{g}\right]\right\} \tag{106}$$

Figure 13 shows the variation of the generalized double force $Y(z)$ along the beam for different values of the ratio h/g, for $g/L = 0.5$ and $D = 5$. Again, the sign of h, i.e., the sign of the pretwist α_0 affects strongly the magnitude of $Y(z)$.

Actuation applications

The examples considered in "Cases studies for various boundary conditions" section suggest some interesting applications, if the pretwisted beam is viewed as a thermally activated actuator. The temperature change along the beam leads to an interplay between the axial force N and the torsional moment T and a coupling between axial and rotational deformation. We recall Eq. (59), which is repeated below and shows that the rotation $\phi(z)$ of the cross sections in the pretwisted beam is directly related to the axial displacement $w_1(z)$:

$$\phi(z) = -\frac{A}{\alpha_0\,S}\left\{-w_1(z) - \alpha\int \Delta\theta(z)\,\mathrm{d}z + \frac{1}{E\,A}\int\left[\int p_z(z)\,\mathrm{d}z\right]\mathrm{d}z\right\} + c_5\frac{z}{L} + c_6 \tag{107}$$

We consider again the example of Case 4 in the previous "Cases studies for various boundary conditions" section, where the pretwisted beam is viewed now as an actuator. The beam is fully constrained at $z = 0\left(w_1(0) = 0,\ \frac{dw_1}{dz}\Big|_{z=0} = 0\right)$ and the generalized loads vanish at the other end ($P(L) = 0$, $Y(L) = 0$). Taking into account Eqs. (72) and (73), which determine the generalized loads, we conclude that the generalized axial load-free condition $P = 0$ at $z = L$ can be achieved by either setting $\bar{N} = 0$ and $\bar{T} = 0$ at that end or, if there is pretwist ($\alpha_0 \neq 0$), by choosing $\bar{N}$ and $\bar{T}$ so that the condition

$$\bar{T} = \frac{1}{\alpha_0\,S}\left(\frac{G}{E}J + \alpha_0^2\,K\right)\bar{N}$$

is satisfied. Last equation suggests a drilling type of action of the pretwisted beam which is at the boundary.

Similarly, the condition $Y = 0$ at $z = L$ can be achieved if the condition

$$h\,\bar{N} + \frac{\alpha_0\,S\,E}{c^2\,J\,G}\bar{B} = 0$$

is satisfied. The displacement at $z = L$ now takes the value

$$w_1(L) = -\alpha\,\Delta\theta_0\,g\,\frac{h + \left[-h + \left(1+\frac{D}{2}\right)L\right]\cosh\frac{L}{g} + \left[-g + \left(1+\frac{D}{2}\right)\frac{h}{g}L\right]\sinh\frac{L}{g}}{g\,\cosh\frac{L}{g} + h\,\sinh\frac{L}{g}} \tag{108}$$

If we also constrain the beam so that $\phi(0) = 0$ and $\frac{d\phi}{dz}\Big|_{z=0} = 0$, Eq. (76) leads to the conclusion that

$$\phi(L) = -\alpha\,\Delta\theta_0\,\frac{A\,g}{\alpha_0\,S}\,\frac{h + (L-h)\cosh\frac{L}{g} + \left(\frac{h}{g}L - g\right)\sinh\frac{L}{g}}{g\cosh\frac{L}{g} + h\sinh\frac{L}{g}} \tag{109}$$

If the cross section of the beam has one axis of symmetry, then $h = 0$ and the above equations simplify to

$$w_1(L)|_{h=0} = \alpha\,\Delta\theta_0\left[\left(1+\frac{D}{2}\right)L - g\tanh\frac{L}{g}\right] \tag{110}$$

$$\phi(L)|_{h=0} = \alpha\,\Delta\theta_0 \frac{A}{\alpha_0\, S}\left(g \tanh\frac{L}{g} - L\right) \tag{111}$$

Equations (108)–(111) show that using the appropriate magnitude of $\Delta\theta_0$ and choosing the geometrical characteristics and the pretwist of the beam, we can control the axial displacement and rotation of the actuator at the free end at $z = L$.

The value of the generalized double force at the fixed end at $z = 0$ is

$$\bar{Y} = h\,\bar{N} + \frac{\alpha_0\, S}{c^2\, J}\frac{E}{G}\bar{B}$$

The magnitude of the bimoment $\bar{B}$ depends on the shape of the cross section and takes substantial values at thin-walled beams. However, if the cross section is such that $\bar{B} \simeq 0$, then

$$\bar{Y} \simeq h\,\bar{N}$$

Since the sign of h depends on the sign of the pretwist, last equation shows that the signs of $\bar{Y}$ and $\bar{N}$ may be different. The generalized double force $\bar{Y}$ is a measure for the warping resistance of the beam's cross section due to torsion. A large value of $|\bar{Y}|$ at $z = 0$ would indicate high local stressing due to the restriction of warping at that end. This stressing will be transmitted from the contacting area between the beam and its supports. The development of such dipolar forces can be utilized together with the torque $\bar{T}$ to act as an effective microdrilling device.

The model of the pretwisted beam under thermal loading can be useful in explaining bio-systems, such as the bacteriophages (Prescott [27]). Bacteriophage is a virus that infects and replicates within a bacterium and can serve as an antibacterial agent treating bacterial infection. Myovirus bacteriophages bind on a bacterial cell and use a cylindrical sheath surrounding a tubular core to puncture the membrane of the cell to inject their genetic material [28]. The sheath is very like the pretwisted beam we have presented in this work. It is made of three polypeptide chains that wind up to form prisms with a left-handed pretwist in their initial configuration. The sheath acts us a cell-puncturing device through a well-documented microdrilling motion that is triggered by chemical reactions. These reactions have the mechanical equivalent of the thermal loading that has been described in the present work. More details of this problem will be addressed in future publications.

Closure-concluding remarks

In the present work, a beam with an initial twist subjected to thermal loads is analyzed using a classical structural approach. The results of the analysis, compared to the results of the one-dimensional gradient thermoelasticity, indicate an interesting analogy between the two approaches. This analogy suggests that the microstructural length-scale parameters of the gradient thermoelastic theory can be related directly to material and geometrical aspects of the continuum as well as the amount of pretwist, providing a physical insight of the gradient theory formulation. The proposed formulation was used to analyze several problems of pretwisted beams under various boundary conditions. These examples demonstrate that the interplay between the generalized loads through temperature variations renders the beam into a thermally activated actuator.

Appendix

Figure 14 (Kordolemis et al. [20]) provides a table with the various geometric constants K, J_ω, J, S, R for several cross sections. The table includes also the values of g and h.

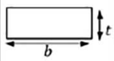
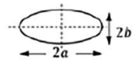
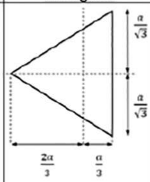
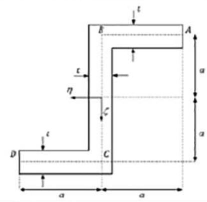
Geometric property	cross section type: thin wall	elliptic	triangular	Z - section
				
$A[m^2]$	bt	πab	$a^2/\sqrt{3}$	$4at$
$\omega^*[m^2]$	$-\zeta_1\zeta_2$	$-\frac{a^2-b^2}{a^2+b^2}\zeta_1\zeta_2$	$\frac{1}{2a}\left(\zeta_2^3-3\zeta_1^2\zeta_2\right)$	$\begin{cases} -a\left(\frac{a}{4}+\zeta_2\right)-\underline{\zeta_2(\zeta_1+a)}(AB) \\ -\frac{a^2}{4}+\underline{\zeta_1\zeta_2}(BC) \\ -a\left(\frac{a}{4}-\zeta_2\right)-\underline{\zeta_2(\zeta_1-a)}(CD) \end{cases}$
$J[m^4]$	$bt^3/3$	$\frac{\pi a^3b^3}{a^2+b^2}$	$\frac{1}{15\sqrt{3}}a^4$	$\frac{4}{3}t^3a$
$K[m^6]$	$b^5t/80$	$\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\frac{\pi ab\left(3a^4+3b^4-2a^2b^2\right)}{24}$	$\frac{7\sqrt{3}}{2430}a^6$	$\frac{62}{15}ta^5$
$S[m^4]$	$b^3t/12$	$\frac{\left(a^2-b^2\right)^2}{a^2+b^2}\frac{\pi ab}{4}$	$\frac{2\sqrt{3}}{135}a^4$	$\frac{10}{3}ta^3$
$R[m^6]$	0	0	0	$\frac{2}{3}ta^5$
$c^2[-]$	$1+\frac{\alpha_0^2}{60}\frac{E}{G}\frac{b^4}{t^2}$	$\frac{\alpha_0^2}{4}\frac{E}{G}\left[\frac{\frac{a^8}{4}-\frac{a^6b^2}{3}+\frac{a^4b^4}{6}-\frac{a^2b^6}{3}+\frac{b^8}{4}}{a^4b^2+a^2b^4}\right]+1$	$1+\frac{\alpha_0^2a^2}{10}\frac{E}{G}$	$1+\frac{61}{60}(\alpha_0 a)^2\left(\frac{a}{t}\right)^2\frac{E}{G}$
$\ell^2[m^2]$	$\frac{b^2}{48}$	$\frac{\left(a^2-b^2\right)^2}{24\left(a^2+b^2\right)}$	$\frac{1}{378}a^2$	$\frac{5}{16}\frac{a^4}{t^2}$
$h[m^2]$	0	0	0	$\frac{a^4}{2t^2\left(1+\frac{61}{60}(a\alpha_0)^2\left(\frac{a}{t}\right)^2\right)}$
$g^2=\frac{E}{G}\frac{\ell^2}{c^2}[m^2]$				

Figure 14. Table with the geometrical constants and lengths for various cross sections.

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