Pretwisted Beams in Axial Tension and Torsion: Analogy with Dipolar Gradient Elasticity and Applications to Textile Materials

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Abstract: A technical theory for pretwisted beams that accounts for variable twist (constrained warping) is based on an approximate displacement field that is defined completely in terms of two unknown functions: the axial displacement $w_1(z)$ and the rotation $\phi(z)$ of the cross section about the centroidal axis of the beam, z being the coordinate along the axis of the beam. The primary unknowns are determined by minimizing the potential energy of the beam. The problem is then formulated in terms of $w_1(z)$, and an analogue between the technical theory and a one-dimensional dipolar gradient elasticity model is presented. The application of the theory to textile materials is discussed. **DOI: 10.1061/(ASCE)EM.1943-7889.0000917.**

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Introduction

Advanced textiles are used in numerous technological applications such as airbags, seat belts, parachutes, and body armor vests. The mechanical properties of textiles classify them as very efficient load-carrying components and their low-cost production and easy handling make them very competitive structural materials.

The structural architecture of textiles is characterized by anisotropy, inhomogeneity, and nonlinear aspects, which are difficult to control simultaneously. Several attempts have been made to develop accurate models that account for the main deformation mechanisms of textiles; the works of Boisse et al. (2011) and King et al. (2005) are recent contributions on the subject.

The mechanics of textile materials can be addressed at three different scales: (1) the macroscopic scale, in which the textile is treated as an anisotropic, nonlinear continuum; (2) the mesoscopic scale, in which the overall mechanical behavior of the textile is characterized by the interactions between the yarns; and (3) the microscopic scale, in which interactions between fibers inside the yarns are taken into account. The present study focuses on the microscopic scale, in the sense that the micromechanical parameters of the yarns are considered.

Most yarns are formed by assembling a large number of (several hundred) fibers, which are pretwisted together about the yarn's longitudinal axis (Fig. 1).

Textile composites have no ability to resist bending or compressive loads and are usually subjected to tensile forces, which tend to stretch the fiber in the longitudinal direction. The mechanical behavior of textile fibers is similar to that of a pretwisted prismatic bar that is subjected to an axial force.

The present paper reviews a technical theory for pretwisted beams that accounts for variable twist (nonuniform tension) and show that it can be viewed as a one-dimensional dipolar-gradient elasticity theory. In particular, an analogue between the aforementioned technical theory and a one-dimensional dipolar-gradient elasticity continuum in tension is presented. The application of the theory to textile materials is discussed.

Tension and Torsion of Pretwisted Beams

The problem of tension/torsion of pretwisted beams has been the subject of numerous publications and is now well understood. Chu (1951) was the first to present an engineering approach to the problem for thin-walled beams; he used the so-called helical fiber assumption that stresses are aligned along helical fibers of the beam and showed that the torsional rigidity of bars with elongated sections is increased by the pretwist. Rosen (1980), Hodges (1980), and Krenk (1983b) developed improved technical theories for pretwisted beams by using assumed forms for the displacement fields and the theorem of minimum potential energy (see also Washizu 1964). A review of the available publications up to 1990 on the structural and dynamic behavior of pretwisted rods and beams can be found in Rosen (1991). Numerical techniques for the solution of the problem of pretwisted beams were developed by Krenk and Gunneskov (1981, 1986), Kosmatka (1992) and Jiang and Henshall (2001). The same problem has been addressed from a mathematical elasticity point of view (as opposed to a technical beam theory) by Okubo (1951, 1953, 1954), Knowles and Reissner (1960), Reissner and Wan (1968), Shield (1982), and Krenk (1983a). Chu (1951) and Rosen (1983) presented experimental results that indicate that the engineering approach is quite adequate for pretwisted thin cross sections at low pretwisted angles.

The current study revisits this problem and shows that it can be formulated as a one-dimensional dipolar-gradient elasticity problem. It considers a cylindrical beam with constant and simply connected cross section. The beam is homogeneous and linear, with Young's modulus E and shear modulus G. A Cartesian coordinate

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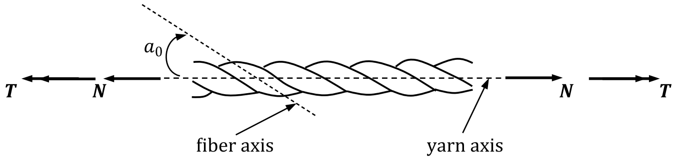

Fig. 1. The angle of pretwist of the fiber is the angle between the fiber axis and the yarn axis

system $Oxyz$ with the z-direction along the centroids of the cross sections and parallel to its generators is introduced. The beam is assumed to be of length L and one of its bases is taken to lie on the xy-plane, where $z = 0$. The beam is pretwisted about the z-axis by a constant amount α_0 per unit length, so that the rotation about the z-axis of the cross section at z is $\phi_0(z) = \alpha_0 z$. On each cross section at z, a local coordinate system $\eta - \zeta$ is introduced by rotating the global $x - y$ axes about the z-axis by an angle $\phi_0(z) = \alpha_0 z$. Then, the analytical expression that describes the boundary of each cross section relative to the $\eta - \zeta$ system is the same for all z. On each cross section at z, the following coordinates have been embedded:

$$\eta(x, y, z) = x\cos(\alpha_0 z) + y\sin(\alpha_0 z)$$
$$\zeta(x, y, z) = -x\sin(\alpha_0 z) + y\cos(\alpha_0 z) \tag{1}$$

so that

$$\frac{\partial\eta}{\partial z} = \alpha_0\zeta, \qquad \frac{\partial\zeta}{\partial z} = -\alpha_0\eta, \qquad \frac{\partial f(\eta,\zeta)}{\partial z} = \alpha_0\left(\zeta\frac{\partial f}{\partial\eta} - \eta\frac{\partial f}{\partial\zeta}\right) \tag{2}$$

where $f(\eta, \zeta)$ = arbitrary function.

No restriction is placed on the amount of pretwist. The beam is assumed to be stress-free in its pretwisted state, and the additional strains and rotations caused by the applied loads are assumed to be infinitesimal, so that linear kinematics are applicable.

In the following, the recent results of Giannakopoulos et al. (2013) for a pretwisted beam in connection to the gradient theories of elasticity are summarized The analysis is based on the formulations of Rosen (1980), Hodges (1980), and Krenk (1983b) and a technical beam theory is used in which the cross sections are assumed to rotate in their own plane without deformation and to move in the axial direction according to the Saint-Venant warping function $\Psi(\eta, \zeta)$ of a similar beam without pretwist. The displacement field accounts for nonuniform twist (restrained warping) and is written in the form [see also Krenk (1983b)]

corrected $\quad u(y, z) = -\phi(z)\, y \qquad v(x, z) = \phi(z)\, x,$

$$w(\eta, \zeta, z) = w_1(z) + \frac{d\phi(z)}{dz}\Psi(\eta, \zeta) \tag{3}$$

where (u, v, w) = displacement components in the (x, y, z) directions; $\phi(z)$ = additional infinitesimal rotation of the pretwisted cross section at z; and $\Psi(\eta, \zeta)$ is normalized so that $\int_A \Psi(\eta, \zeta) d\eta d\zeta = 0$. This displacement field is the same as that used by (1) Rosen (1980), if the twist is uniform (i.e., $d\phi/dz = \text{const.}$); (2) Hodges (1980), if $\phi(z)$ assumed to be small; and (3) Krenk (1983b), if the bending terms he introduced are dropped. The contribution of this work, in comparison with that of Giannakopoulos et al. (2013), is that the present formulation is in terms of axial displacements and displacement gradients. This permits the formulation of a one-dimensional continuum analogue that Giannakopoulos et al. (2013) cannot provide, since their theory relies on the rotation, and rotation gradient degrees of freedom and provide a structural theory of torsion.

The assumed approximate displacement field in Eq. (3) is defined completely in terms of $w_1(z)$ and $\phi(z)$, which are the primary unknowns determined by minimizing the potential energy of the beam. In this approach, the traction-free boundary conditions on the lateral surface of the beam are satisfied only approximately. The assumption that $\Psi(\eta, \zeta)$ in Eq. (3) is the Saint-Venant warping function of the beam without pretwist is known to be accurate for small pretwist: Krenk (1983a) used an asymptotic expansion of the complete set of three-dimensional equations of linear elasticity and showed that, when the pretwist α_0 is small, $\Psi(\eta, \zeta)$ is, to leading order, the usual Saint-Venant warping function; the accuracy of this assumption for finite values of pretwist has been discussed in detail by Liu et al. (2009).

The beam is loaded by axial forces and torsional moments, and the condition of minimum potential energy is written in the form

$$\delta(U - W) = 0 \tag{4}$$

where U = elastic strain energy of the beam; variation δW of external work

$$\delta W = \int_0^L (p_z \delta w_1 + m_z \delta\phi) dz + (N\delta w_1)_0^L + (T\delta\phi)_0^L + \left(-B\frac{d\delta\phi}{dz}\right)_0^L \tag{5}$$

N = axial load; T = torque; $B = -\int_A \sigma_{zz}\Psi dA$, i.e., the bimoment; $p_z = -(dN/dz)$ = distributed axial load per unit length; and $m_z = -(dT/dz)$ = distributed torsional moment per unit length. The resulting governing equations and boundary conditions are (Giannakopoulos et al. 2013)

$$\frac{d^2 w_1(z)}{dz^2} + \frac{\alpha_0 S}{A}\frac{d^2\phi(z)}{dz^2} = -\frac{p_z(z)}{EA} \tag{6}$$

$$-\ell^2\frac{d^4\phi(z)}{dz^4} + \left(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G}\right)\frac{d^2\phi(z)}{dz^2} + \frac{\alpha_0 S}{J}\frac{E}{G}\frac{d^2 w_1(z)}{dz^2} = -\frac{m_z(z)}{GJ} \tag{7}$$

and at the ends of the beam

1. Either the axial displacement

$$w_1 = \bar{w}_1 = \text{known} \tag{8}$$

$$\text{or the axial force } \frac{dw_1}{dz} + \frac{\alpha_0 S}{A}\frac{d\phi}{dz} = \frac{\bar{N}}{EA} = \text{known} \tag{9}$$

2. Either the twist

$$\phi = \bar{\phi} = \text{known} \tag{10}$$

$$\text{or the torque } -\ell^2\frac{d^3\phi}{dz^3} + \left(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G}\right)\frac{d\phi}{dz} + \frac{\alpha_0 S}{J}\frac{E}{G}\frac{dw_1}{dz} = \frac{\bar{T}}{GJ} = \text{known} \tag{11}$$

3. Either the rate of twist

$$\frac{d\phi}{dz} = \bar{\phi}' = \text{known} \tag{12}$$

$$\text{or the bimoment } \ell^2 \frac{d^2\phi}{dz^2} + \ell_1 \frac{d\phi}{dz} = -\frac{\bar{B}}{GJ} = \text{known} \tag{13}$$

where

$$J = \int_A \left(\zeta^2 + \eta^2 + \eta \frac{\partial \Psi}{\partial \zeta} - \zeta \frac{\partial \Psi}{\partial \eta} \right) d\zeta d\eta = \int_A \left[\left(\frac{\partial \Psi}{\partial \zeta} + \eta \right)^2 + \left(\frac{\partial \Psi}{\partial \eta} - \zeta \right)^2 \right] d\zeta d\eta > 0 \tag{14}$$

$$J_\omega = \int_A \Psi^2(\eta, \zeta) d\eta d\zeta \geq 0 \qquad \ell = \sqrt{\frac{EJ_\omega}{GJ}} \geq 0 \qquad \ell_1 = \alpha_0 \frac{ER}{GJ} \tag{15}$$

$$K = \frac{1}{\alpha_0^2} \int_A \left(\frac{\partial \Psi}{\partial z} \right)^2 d\eta d\zeta = \int_A \left(\zeta \frac{\partial \Psi}{\partial \eta} - \eta \frac{\partial \Psi}{\partial \zeta} \right)^2 d\eta d\zeta \geq 0 \tag{16}$$

$$R = \frac{1}{\alpha_0} \int_A \Psi \frac{\partial \Psi}{\partial z} d\eta d\zeta = \int_A \Psi \left[\left(\frac{\partial \Psi}{\partial \eta} \right)^2 + \left(\frac{\partial \Psi}{\partial \zeta} \right)^2 \right] d\eta d\zeta \tag{17}$$

$$S = \frac{1}{\alpha_0} \int_A \frac{\partial \Psi}{\partial z} d\eta d\zeta = \int_A \left[\left(\frac{\partial \Psi}{\partial \eta} \right)^2 + \left(\frac{\partial \Psi}{\partial \zeta} \right)^2 \right] d\eta d\zeta \geq 0 \tag{18}$$

In the definitions above, J is the usual Saint-Venant torsional constant of the cross section and ℓ is a length scale defined in Eq. (15b) as the square root of the ratio of the warping rigidity EJ_ω to the torsional rigidity GJ, and appears naturally in Vlasov's (1961) theory of constrained warping of thin-walled beams. The ℓ-terms are associated with the stresses that develop normal to the cross section because of nonuniform twist ($d\phi/dz \neq$ const.) and appear both in the torsional moment [Eq. (7)] and the boundary condition [Eq. (13)]. Parameter ℓ_1, defined in Eq. (15c) is a length scale proportional to the amount of pretwist α_0, depends on the shape of the cross section, and affects only the bimoment boundary condition [Eq. (13)]. If the cross section has an axis of symmetry, then the parameter R and the length scale ℓ_1 vanish. For example, if the η-axis is an axis of symmetry, then $\Psi(\eta, -\zeta) = -\Psi(\eta, \zeta)$, so that $\partial\Psi/\partial z|_{(\eta,-\zeta)} = \partial\Psi/\partial z|_{(\eta,\zeta)}$, where $\partial\Psi/\partial z$ is defined in Eq. (2c); then, in view of the aforementioned symmetry, $R = 1/\alpha_0 \int_A \Psi(\partial\Psi/\partial z) d\eta d\zeta = 0$.

The ℓ-terms in Eqs. (6)–(13) vanish in the case of uniform torsion ($d\phi/dz =$ const.) and the ℓ_1-terms vanish when there is no pretwist ($\alpha_0 = 0$).

Eqs. (6) and (7) can be integrated once to yield

$$\frac{dw_1(z)}{dz} + \frac{\alpha_0 S}{A} \frac{d\phi(z)}{dz} = \frac{N(z)}{EA} \tag{19}$$

$$-\ell^2 \frac{d^3\phi(z)}{dz^3} + \left(1 + \frac{\alpha_0^2 K}{J} \frac{E}{G} \right) \frac{d\phi(z)}{dz} + \frac{\alpha_0 S}{J} \frac{E}{G} \frac{dw_1(z)}{dz} = \frac{T(z)}{GJ} \tag{20}$$

Note that the classical problem of uniform twist ($d\phi/dz =$ const.) is recovered by letting $\ell \to 0$ in Eqs. (6)–(13) and (19)–(20). However, since ℓ^2 multiplies the highest derivatives of $\phi(z)$ in the governing Eqs. (7) and (20) and in the boundary conditions [Eqs. (11) and (13)], the case of a small nonzero ℓ is a singular perturbation of the classical problem in which $\ell = 0$, and the limit $\ell \to 0$ of the solutions should be taken with care.

Giannakopoulos et al. (2013) eliminated $w_1(z)$ between Eqs. (6) and (7) and formulated the problem in terms of $\phi(z)$. Here an alternative approach is taken and the problem is formulated in terms of $w_1(z)$ as follows. For $\alpha_0 S \neq 0$, $\phi(z)$ can be eliminated from Eqs. (6)–(13) and the boundary value problem defined in terms of $w_1(z)$:

$$\left(\frac{\ell^2}{c^2} \frac{d^2}{dz^2} - 1 \right) \frac{d^2 w_1(z)}{dz^2} = \frac{q(z)}{EA} \tag{21}$$

and at the ends of the beam

$$w_1 = \bar{w}_1 = \text{known} \quad \text{or} \quad -\frac{\ell^2}{c^2} \frac{d^3 w_1}{dz^3} + \frac{dw_1}{dz} = \frac{\bar{P}}{EA} = \text{known} \tag{22}$$

$$\frac{dw_1}{dz} = \bar{w}'_1 = \text{known} \quad \text{or} \quad \frac{\ell^2}{c^2} \frac{d^2 w_1}{dz^2} + \frac{\ell_1}{c^2} \frac{dw_1}{dz} = \frac{\bar{Y}}{EA} = \text{known} \tag{23}$$

where

$$q(z) = \frac{1}{c^2} \left\{ \left(1 - \ell^2 \frac{d^2}{dz^2} \right) p_z(z) + \frac{E}{G} \left[\frac{\alpha_0^2 K}{J} p_z(z) - \frac{\alpha_0 S}{J} m_z(z) \right] \right\} \tag{24}$$

$$c(\alpha_0) = \sqrt{1 + \frac{\alpha_0^2}{J} \frac{E}{G} \left(K - \frac{S^2}{A} \right)} \qquad c(0) = 1 \tag{25}$$

$$\bar{P} = \frac{1}{c^2} \left[\bar{N} + \ell^2 \frac{dp_z}{dz} + \frac{E}{G} \left(\frac{\alpha_0^2 K}{J} \bar{N} - \frac{\alpha_0 S}{J} \bar{T} \right) \right], \qquad \bar{Y} = \frac{1}{c^2} \left(-\ell^2 p_z + \ell_1 \bar{N} + \frac{\alpha_0 S}{J} \frac{E}{G} \bar{B} \right) \tag{26}$$

$$\bar{w}'_1 = \frac{\bar{N}}{EA} - \frac{\alpha_0 S}{A} \bar{\phi}' \tag{27}$$

The quantities q, $\bar{P}$, and $\bar{Y}$ are generalized loads and are defined in terms of the classical loads p_z, m_z, $\bar{N}$, $\bar{T}$, and $\bar{B}$ in Eqs. (24) and (26). Similarly, the boundary values $\bar{w}'_1$ are defined in terms of $\bar{N}$ and $\bar{\phi}'$ in Eq. (27). Note that the difference between the actual axial load $\bar{N}$ and the generalized load $\bar{P}$ is due to pretwist ($\alpha_0 \neq 0$) and nonuniform twist ($\ell \neq 0$); in the classical case where $\alpha_0 = 0$ and $\ell = 0$, $\bar{P} = \bar{N}$, $q = p_z$, and $\bar{Y} = 0$.

Once $w_1(z)$ is determined from Eqs. (21)–(23), the solution is completed with the determination of $\phi(z)$ from Eq. (19), i.e.,

$$\frac{d\phi(z)}{dz} = \frac{A}{\alpha_0 S} \left[\frac{N(z)}{EA} - \frac{dw_1(z)}{dz} \right] \tag{28}$$

and the boundary conditions [Eq. (10) or Eq. (11)].

This section is concluded by emphasizing that the formulation of the problem in terms of $w_1(z)$ is possible only when $\alpha_0 S \neq 0$, i.e., when pretwist is present.

Analogy of the One-Dimensional Dipolar Gradient Elasticity Model

Tsepoura et al. (2002) presented a one-dimensional gradient elasticity theory for a bar with an elastic-strain energy-density function of the form

$$\bar{U}(\varepsilon, \kappa) = \frac{E}{2}(\varepsilon^2 + g^2\kappa^2 + 2m\varepsilon\kappa) \tag{29}$$

where $w_1(z)$ = axial displacement; $\varepsilon = dw_1/dz$ = axial strain in a bar; $\kappa = d\varepsilon/dz = d^2w_1/dz^2$ = strain gradient; and g, m = material lengths. The corresponding axial stress σ and double stress μ are determined from $\bar{U}$

$$\sigma = \frac{\partial \bar{U}}{\partial \varepsilon} = E(\varepsilon + m\kappa) = E\left(\frac{dw_1}{dz} + m\frac{d^2w_1}{dz^2}\right) \tag{30}$$

$$\mu = \frac{\partial \bar{U}}{\partial \kappa} = E(m\varepsilon + g^2\kappa) = E\left(m\frac{dw_1}{dz} + g^2\frac{d^2w_1}{dz^2}\right) \tag{31}$$

so that $\bar{U}$ can be written also in the form (Casal 1961; Vardoulakis et al. 1996)

$$\bar{U} = \frac{1}{2}(\sigma\varepsilon + \mu\kappa) \tag{32}$$

Papargyri-Beskou et al. (2009) used the same one-dimensional dipolar-gradient elasticity model to study analytically the microstructural effects of wave dispersion and established analogies with axial (Love) beams enriched with lateral inertia effects, Timoshenko beams, and Mindlin plates enriched with microinertia terms.

Tsepoura et al. (2002) showed that the governing equation and boundary conditions for $w_1(z)$ resulting from the variational statement $\delta U = \delta W$ are

$$\left(g^2\frac{d^2}{dz^2} - 1\right)\frac{d^2w_1}{dz^2} = \frac{q}{EA} \tag{33}$$

and at the ends of the bar

$$w_1 = \bar{w}_1 = \text{known} \quad \text{or} \quad -g^2\frac{d^3w_1}{dz^3} + \frac{dw_1}{dz} = \frac{\bar{P}}{EA} = \text{known} \tag{34}$$

$$\frac{dw_1}{dz} = \bar{w}_1' = \text{known} \quad \text{or} \quad g^2\frac{d^2w_1}{dz^2} + m\frac{dw_1}{dz} = \frac{\bar{Y}}{EA} = \text{known} \tag{35}$$

Eqs. (33)–(35) are identical to Eqs. (21)–(23), provided the following substitutions are made:

$$g^2 \leftrightarrow \frac{\ell^2}{c^2} = \frac{EJ_\omega}{GJ + \alpha_0^2E(K - \frac{S^2}{A})}$$
$$m \leftrightarrow \frac{\ell_1}{c^2} = \frac{\alpha_0 ER}{GJ + \alpha_0^2E(K - \frac{S^2}{A})} \tag{36}$$

Therefore, the one-dimensional gradient elasticity theory can be thought of as a homogenization theory that accounts for the effects of constrained warping and pretwist. The terms that involve g^2 and the highest derivatives of w_1 in Eqs. (33)–(35) are due to nonhomogeneous twist and vanish when $d\phi/dz = \text{const.}$ along the beam. The effects of pretwist influence the solution through the constants m and $c(\alpha_0)$. Note that the material length g decreases as the amount of pretwist $|\alpha_0|$ increases, and that the sign of ℓ_1 (and m) depends on the sign (direction) of pretwist α_0.

The m-term in Eq. (29) is often attributed to surface energy (Casal 1961, 1963, 1972; Vardoulakis and Sulem 1995; Vardoulakis et al. 1996). According to the analogue, Eq. (36), the surface length m that enters the boundary conditions of the gradient theory is attributed to the lack of symmetry of the cross section; cross sections with at least one axis of symmetry have zero surface length $(m = 0)$, because the parameter R defined in Eq. (17) vanishes.

It can be shown that

$$-c(\alpha_0) \le \frac{\alpha_0 ER}{\sqrt{(GJ)(EJ_\omega)}} \le c(\alpha_0) \tag{37}$$

which implies that $-g \le m \le g$, so that the strain energy density defined in Eq. (29) is positive definite (Georgiadis et al. 2000, 2004).

Fig. 2 gives the complete analogy of the pretwisted problem and the one-dimensional dipolar gradient elasticity model.

When the analogy is used, the results should be interpreted with care. In the one-dimensional gradient elasticity model, the axial strain ε_{zz} is uniform on every cross section and is defined simply as

$$\varepsilon_{zz}(z) = \frac{dw_1(z)}{dz} \tag{38}$$

whereas in the technical theory of pretwisted beams, the axial strain is defined using Eq. (3) for $w(x, y, z)$ and takes the value

$$\varepsilon_{zz}(x,y,z) = \frac{\partial w(x,y,z)}{\partial z} = \frac{dw_1(z)}{dz} + \frac{d^2\phi(z)}{dz^2}\Psi(x,y,z) + \frac{d\phi(z)}{dz}\frac{\partial \Psi(x,y,z)}{\partial z} \tag{39}$$

The additional terms in the equation above are due to the explicit consideration of the nonuniform twist $(d^2\phi/dz^2 \neq 0)$ and the pretwist $(\partial\Psi/\partial z \neq 0)$. The average strain over a cross section resulting from Eq. (39) is

$$\bar{\varepsilon}_{zz}(z) = \frac{1}{A}\int_A \varepsilon_{zz}(\eta, \zeta, z)d\eta d\zeta = \frac{dw_1(z)}{dz} + \frac{\alpha_0 S}{A}\frac{d\phi(z)}{dz} \tag{40}$$

Taking into account Eq. (19), it can be concluded that the above equation can be also written as

$$\bar{\varepsilon}_{zz}(z) = \frac{N(z)}{EA} \equiv \varepsilon_N(z) \tag{41}$$

i.e., in the technical theory of pretwisted beams, the average axial strain on a cross section is determined from the usual expression of the classical theory in terms of the actual axial load N.

This section is concluded with a brief discussion of the "surface energy" m-term in the expression for the elastic strain energy density.

As discussed by Vardoulakis and Sulem (1995), Casal's (1961) proposal for the surface energy term is not consistent with Mindlin's (1964) linear, isotropic gradient elasticity theory. Vardoulakis and Sulem (1995) introduce a surface energy term together with a constant characteristic director vector in the context of anisotropic gradient elasticity. An alternative view, consistent with Casal's (1963) original proposal, is presented here. Let $\boldsymbol{\varepsilon}$ be the infinitesimal strain tensor and define the third-order strain gradient tensor $\boldsymbol{\kappa} = \nabla\boldsymbol{\varepsilon}$. If one requires that the elastic strain energy density $\bar{U}$

variable	pretwisted beam	1-D gradient elasticity
axial displacement	$w_1(z)$	$w_1(z)$
volume material length	$\ell=\sqrt{\frac{EJ_\omega}{GJ}}$	$g=\frac{\ell}{c}$
surface material length	$\ell_1=a_0\frac{ER}{GJ}$	$m=\frac{\ell_1}{c^2}$
body force	$\frac{1}{c^2}\left[\left(1-\ell^2\frac{d^2}{dz^2}\right)p_z(z)+\frac{\alpha_0^2 EK}{GJ}p_z(z)-\frac{\alpha_0 ES}{GJ}m_z(z)\right]$	$q(z)$
traction like boundary force	$\frac{1}{c^2}\left(\bar{N}+\ell^2\frac{dp_z}{dz}+\frac{\alpha_0^2 EK}{GJ}\bar{N}-\frac{\alpha_0 ES}{GJ}\bar{T}\right)$	$\bar{P}$
couple like boundary force	$\frac{1}{c^2}\left(-\ell^2 p_z+\ell_1\bar{N}+\frac{\alpha_0 ES}{GJ}\bar{B}\right)$	$\bar{Y}$
$c=\sqrt{1+\frac{\alpha_0^2 E}{GJ}\left(K-\frac{S^2}{A}\right)}>1$		

Fig. 2. Analogy between tension of a pretwisted beam and one-dimensional dipolar gradient elasticity

of a linear elastic material is an isotropic function of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, i.e., a function invariant relative to the full orthogonal group, then $\bar{U}$ involves seven material constants and must be of the form (Mindlin and Eshel 1968; dell'Isola et al. 2009)

$$\bar{U}^{\text{isotropic}}(\boldsymbol{\varepsilon},\boldsymbol{\kappa})=\frac{1}{2}(L_{ijkm}\varepsilon_{ij}\varepsilon_{km}+G_{ijkmpq}\kappa_{kij}\kappa_{qmp}) \tag{42}$$

where L = usual fourth-order isotropic elasticity tensor, i.e.,

$$L_{ijkl}=\kappa\delta_{ij}\delta_{km}+G(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk}),\quad \kappa=\frac{G(E-2G)}{3G-E} \tag{43}$$

and

$$\begin{aligned}G_{ijkmpq}&=c_2[\delta_{ij}(\delta_{km}\delta_{pq}+\delta_{kp}\delta_{mq})+\delta_{mp}(\delta_{ik}\delta_{jq}+\delta_{iq}\delta_{jk})]\\&\quad+c_3\delta_{ij}\delta_{kq}\delta_{mp}+c_5[\delta_{ik}(\delta_{jm}\delta_{pq}+\delta_{jp}\delta_{mq})\\&\quad+\delta_{jk}(\delta_{im}\delta_{pq}+\delta_{ip}\delta_{mq})]+c_{11}\,\delta_{kq}(\delta_{im}\delta_{jp}+\delta_{ip}\delta_{jm})\\&\quad+c_{15}[\delta_{jq}(\delta_{im}\delta_{kp}+\delta_{ip}\delta_{km})+\delta_{iq}(\delta_{jm}\delta_{kp}+\delta_{jp}\delta_{km})]\end{aligned} \tag{44}$$

$(c_2,c_3,c_5,c_{11},c_{15})$ being material constants. In Eq. (42), there are no $\boldsymbol{\varepsilon}-\boldsymbol{\kappa}$ crossterms and surface energy is excluded.

However, if one requires that the elastic strain energy of the isotropic material is a hemitropic function of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, i.e., a function invariant relative to the proper orthogonal group, as suggested originally by Casal (1963), then $\bar{U}$ involves eight material constants and must be of the form (Casal 1963; Suiker and Chang 2000; dell'Isola et al. 2009)

$$\begin{aligned}\bar{U}^{\text{hemitropic}}(\boldsymbol{\varepsilon},\boldsymbol{\kappa})&=\bar{U}^{\text{isotropic}}(\boldsymbol{\varepsilon},\boldsymbol{\kappa})+c_8(e_{ikm}\delta_{jp}+e_{ikp}\delta_{jm}\\&\quad+e_{jkm}\delta_{ip}+e_{jkp}\delta_{im})\kappa_{kij}\varepsilon_{mp}\end{aligned} \tag{45}$$

where e_{ijk} = Levi–Civita alternator and c_8 = additional material length.

In the case of the pretwisted beam, the only nonzero strain components corresponding to the displacement field [Eq. (3)] are ε_{zx}, ε_{zy}, and ε_{zz}, and the hemitropic strain energy density takes the form

$$\begin{aligned}\bar{U}^{\text{hemitropic}}&=\bar{U}^{\text{isotropic}}+4c_8[\varepsilon_{zx}(\kappa_{zzy}-\kappa_{yzz})\\&\quad+\varepsilon_{zy}(\kappa_{xzz}-\kappa_{zzx})+\varepsilon_{zz}(\kappa_{yzx}-\kappa_{xzy})]\\&=\bar{U}^{\text{isotropic}}+4c_8[\varepsilon_{zx}(\varepsilon_{zy,z}-\varepsilon_{zz,y})\\&\quad+\varepsilon_{zy}(\varepsilon_{zz,x}-\varepsilon_{zx,z})+\varepsilon_{zz}(\varepsilon_{zx,y}-\varepsilon_{zy,x})]\end{aligned} \tag{46}$$

Note that the hemitropic form of the elastic strain energy density does not include terms such as $\varepsilon_{zz}\kappa_{zzz}$, which are used in Eq. (29), but does allow for $\boldsymbol{\varepsilon}-\boldsymbol{\kappa}$ crossterms that involve different strain components. A strain energy density of the form in Eq. (46) may be appropriate for materials that do not possess reflective symmetries; an example could be the textile yarns discussed in Section "Strength of Textile Yarns," which are viewed as macroscopically isotropic but are not necessarily symmetric with respect to planes perpendicular to their axis.

General Solution

The general solution of the governing Eq. (21) is

$$\begin{aligned}w_1(z)&=A_1+A_2z+A_3\cosh\frac{z}{\ell/c}+A_4\sinh\frac{z}{\ell/c}\\&\quad-\frac{\ell/c}{EA}\int_0^z\left(\frac{z-\xi}{\ell/c}-\sinh\frac{z-\xi}{\ell/c}\right)q(\xi)d\xi\end{aligned} \tag{47}$$

where A_1, A_2, A_3, and A_4 = arbitrary constants determined from the boundary conditions.

To get some insight on the nature of the above solution, consider the special case in which $p_z = 0$ and $m_z = 0$, so that the beam is loaded with a constant axial force $N(z) = \bar{N}$ and a constant torque $T(z) = \bar{T}$. Then

$$q = 0, \qquad \bar{P} = \frac{1}{c^2}\left[\bar{N} + \frac{E}{G}\left(\frac{\alpha_0^2 K}{J}\bar{N} - \frac{\alpha_0 S}{J}\bar{T}\right)\right],$$
$$\bar{Y} = \frac{1}{c^2}\left(\ell_1 \bar{N} + \frac{\alpha_0 S}{J}\frac{E}{G}\bar{B}\right) \tag{48}$$

and the general solution reduces to

$$w_1(z) = A_1 + A_2 z + A_3 \cosh\frac{z}{\ell/c} + A_4 \sinh\frac{z}{\ell/c} \tag{49}$$

Because $N(z) = \text{const.} \equiv \bar{N}$, Eq. (28) yields

$$\phi(z) = \frac{A}{\alpha_0 S}[\varepsilon_N z - w_1(z)] + A_5, \qquad \varepsilon_N \equiv \frac{\bar{N}}{EA} \tag{50}$$

where A_5 = constant.

Assume that the beam is fixed at $z = 0$, i.e.,

$$w_1(0) = 0, \qquad \phi(0) = 0 \qquad \text{and} \qquad \phi'(0) = 0 \tag{51}$$

and loaded at $z = L$ with an axial force $\bar{N}$, a torque $\bar{T}$, and a bimoment $\bar{B}$. In view of Eq. (27), the boundary conditions at $z = 0$ can be written in the form

$$w_1(0) = 0, \qquad \phi(0) = 0, \quad \text{and} \quad \bar{w}_1'(0) = \varepsilon_N \tag{52}$$

At the other end $z = L$, the loads $\bar{N}$, $\bar{T}$, and $\bar{B}$ define the generalized loads $\bar{P}$ and $\bar{Y}$ through Eq. (48) and the boundary conditions become

$$-\frac{\ell^2}{c^2}\frac{d^3 w_1}{dz^3} + \frac{dw_1}{dz} = \frac{\bar{P}}{EA} \equiv \varepsilon_P \quad \text{and} \quad \frac{\ell^2}{c^2}\frac{d^2 w_1}{dz^2} + \frac{\ell_1}{c^2}\frac{dw_1}{dz} = \frac{\bar{Y}}{EA} \equiv \varepsilon_Y L \tag{53}$$

The boundary conditions Eqs. (52) and (53) define the constants in the solution as follows:

$$A_1 = -A_3 = \frac{\ell}{c}\frac{-\varepsilon_Y L + \varepsilon_P \frac{\ell_1}{c^2} + (\varepsilon_N - \varepsilon_P)\left(\frac{\ell_1}{c^2}\cosh\frac{L}{\ell/c} + \frac{\ell}{c}\sinh\frac{L}{\ell/c}\right)}{\frac{\ell}{c}\cosh\frac{L}{\ell/c} + \frac{\ell_1}{c^2}\sinh\frac{L}{\ell/c}} \tag{54}$$

$$A_2 = \varepsilon_P, \qquad A_4 = \frac{\ell}{c}(\varepsilon_N - \varepsilon_P), \qquad A_5 = 0 \tag{55}$$

If the cross section is assumed to have an axis of symmetry, so that $R = 0$ and $\ell_1 = 0$, and that the loads are applied at $z = L$ in such a way that $\bar{B} = 0$, then $\bar{Y} = 0$, $\varepsilon_Y = 0$, and

$$A_1 = -A_3 = \frac{\ell}{c}(\varepsilon_N - \varepsilon_P)\tanh\frac{L}{\ell/c} \tag{56}$$

so that the solution simplifies to

$$\frac{w_1(z)}{L} = \frac{z}{L}\varepsilon_N + (\varepsilon_P - \varepsilon_N)F\left(\frac{z}{L}, \frac{\ell}{cL}\right),$$
$$\frac{dw_1(z)}{dz} = \varepsilon_N + (\varepsilon_P - \varepsilon_N)H\left(\frac{z}{L}, \frac{\ell}{cL}\right) \tag{57}$$

$$\phi(z) = -\frac{AL}{S}\frac{\varepsilon_P - \varepsilon_N}{\alpha_0}F\left(\frac{z}{L}, \frac{\ell}{cL}\right) \tag{58}$$

where

$$F\left(\frac{z}{L}, \frac{\ell}{cL}\right) = \frac{z}{L} - \frac{\ell/c}{L}\frac{\sinh\frac{L}{\ell/c} - \sinh\frac{L-z}{\ell/c}}{\cosh\frac{L}{\ell/c}},$$
$$H\left(\frac{z}{L}, \frac{\ell}{cL}\right) = L\frac{\partial F}{\partial z} = 1 - \frac{\cosh\frac{L-z}{\ell/c}}{\cosh\frac{L}{\ell/c}} \tag{59}$$

The first terms in Eq. (57) correspond to the classical solution ($\alpha_0 = 0, \ell = 0$), i.e., $w_1(z) = \varepsilon_N z$. The effects of pretwist enter the solution though ε_P and the constant $c(\alpha_0)$ defined in Eq. (25); the effects of nonuniform twist enter the solution through ℓ in the dimensionless functions F and H. The functions $F[(z/L), (\ell/cL)]$ and $H(z/L), (\ell/cL)$ take positive values for $0 < z < L$ and $\ell > 0$ and are decreasing with ℓ. Therefore, Eq. (57) show that the influence of pretwist on the axial displacement $w_1(z)$ and on the derivative $dw_1(z)/dz$ depend on the sign of the difference $\varepsilon_P - \varepsilon_N = (\bar{P} - \bar{N})/EA$, which vanishes when there is no pretwist ($\alpha_0 = 0$). Note that ε_N and ε_P are the traditional axial strains that correspond to the actual and generalized loads $\bar{N}$ and $\bar{P}$, respectively. In view of Eq. (52b), the difference $\varepsilon_P - \varepsilon_N$ can be written in the form

$$\varepsilon_P - \varepsilon_N = \frac{\bar{P} - \bar{N}}{EA} = \frac{1}{c^2(\alpha_0)}\frac{S}{A}\frac{1}{GJ}\left(\frac{\alpha_0^2 S}{A}\bar{N} - \alpha_0 \bar{T}\right)$$
$$c^2(\alpha_0) = 1 + \frac{\alpha_0^2}{J}\frac{E}{G}\left(K - \frac{S^2}{A}\right) \tag{60}$$

Therefore, if the applied loads and the pretwist are such that

$$\bar{P} > \bar{N} \quad \text{or} \quad \frac{\alpha_0^2 S}{A}\bar{N} > \alpha_0 \bar{T} \tag{61}$$

then the axial displacement $w_1(z)$ and the derivative $[dw_1(z)]/dz$ are higher in the pretwisted beam. The opposite is true when the inequalities [Eq. (61)] are reversed.

Also, because $F[(z/L), (\ell/cL)]$ and $H(z/L), (\ell/cL)$ are decreasing functions of ℓ, the higher the material length ℓ, the less pronounced the effects of pretwist become. Figs. 3 and 4 show the variation of the normalized axial displacement $[w_1(z)]/L\varepsilon_N$ and the normalized axial displacement gradient $[dw_1(z)/dz]/\varepsilon_N$ along the beam for various values of ℓ/cL and $(\varepsilon_P - \varepsilon_N)/\varepsilon_N = (\bar{P} - \bar{N})/\bar{N} = \pm 0.6$.

Strength of Textile Yarns

Textile materials are composites that have a microstructure in the form of connected yarns. The yarns can be viewed as pretwisted beams that are loaded mainly in tension. Therefore, the equations governing their mechanical behavior have a close analogy with those of dipolar gradient elasticity with microstructural lengths and generalized loads that relate to their geometry, amount of pretwist, and elastic constants. The stiffness and strength of textiles are known to exhibit size phenomena that can be rationalized by using dipolar gradient elasticity.

In the following, the stiffness and strength of textile yarns in the context of gradient elasticity are considered (Fig. 1). In a uniaxial tension test, a yarn is fixed at one end and an axial load $\bar{N}$ or a displacement $w_1(L)$ is applied at the other. The usual interpretation of the test data is to relate the stress $\sigma = \bar{N}/A$ to the macroscopic

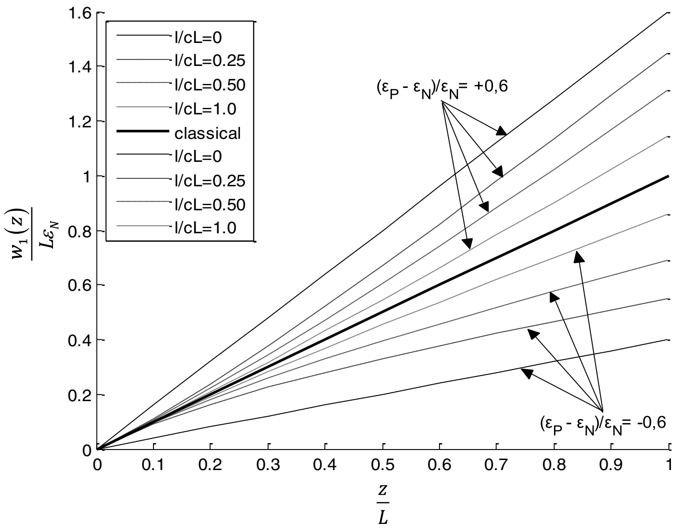

Fig. 3. Normalized axial displacement $w_1(z)/L\varepsilon_N$ along the beam for various values of ℓ/cL, for $(\varepsilon_P-\varepsilon_N)/\varepsilon_N=(\bar{P}-\bar{N})/\bar{N}=+0.6$ (above the classical solution) and -0.6

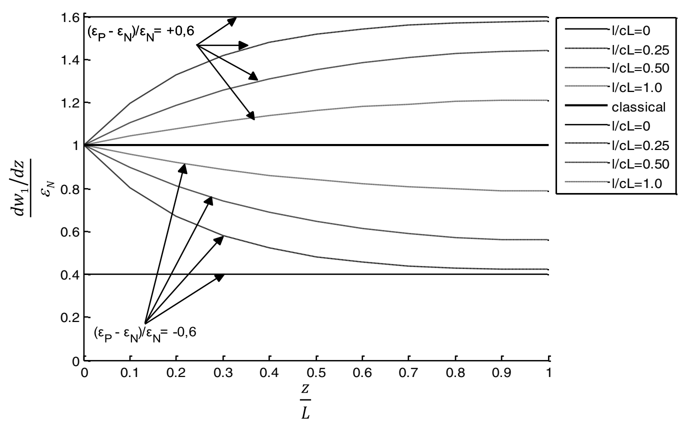

Fig. 4. Normalized axial displacement $[dw_1(z)/dz]/\varepsilon_N$ along the beam for various values of ℓ/cL, for $(\varepsilon_P-\varepsilon_N)/\varepsilon_N=(\bar{P}-\bar{N})/\bar{N}=+0.6$ and -0.6

strain $e=[w_1(L)]/L$. If the yarn has no pretwist ($\alpha_0=0$), the relationship is of the form

$$\frac{\bar{N}}{EA}=\frac{w_1(L)}{L}\quad\text{or}\quad\frac{\sigma}{E}=e \tag{62}$$

which defines experimentally the Young's modulus E. When there is pretwist, Eq. (57) can be used to find

$$\frac{w_1(L)}{L}=\frac{\bar{N}}{EA}\left[1+\left(\frac{\bar{P}}{\bar{N}}-1\right)F\left(1,\frac{\ell}{cL}\right)\right] \tag{63}$$

which leads to

$$\frac{\sigma}{\bar{E}}=\bar{\varepsilon}\quad\text{with }\bar{E}=\frac{E}{1+\frac{1}{c^2}\frac{\alpha_0^2S^2}{AJ}\frac{E}{G}\left(1-\frac{\ell}{cL}\tanh\frac{\ell}{cL}\right)}<E \tag{64}$$

The last equation defines an apparent Young's modulus $\bar{E}$, which is calculated based on the end displacement $w_1(L)$ and ignores the fact that the strain distribution is nonuniform along the yarn. The yarn appears to be softer, in the sense that $\bar{E}<E$, because the value of $w_1(L)$ increases in this case because of untwisting. Indeed, experimental evidence suggests this apparent response (Hearle et al. 1969). The stiffer response of short lengths of textiles can be found in experimental studies of drape (Hearle et al. 1969).

For assessing the strength of textile yarns, it is reasonable to assume that the yarn fails at a critical intrinsic strain level in the axial direction. The solution developed in Section "General Solution" for a pretwisted yarn fixed at $z=0$ and loaded by an axial force $\bar{N}$ and a torque $\bar{T}$ at $z=L$ shows that the axial displacement field is such that

$$w_1(z)=\frac{\bar{N}}{EA}z+L\frac{\bar{P}-\bar{N}}{EA}F\left(\frac{z}{L},\frac{\ell}{cL}\right)\quad\text{and}$$

$$\frac{dw_1(z)}{dz}=\frac{\bar{N}}{EA}+\frac{\bar{P}-\bar{N}}{EA}H\left(\frac{z}{L},\frac{\ell}{cL}\right) \tag{65}$$

where the dimensionless functions F and H are defined in Eq. (59) and take positive values for $0<z<L$ and $(\ell/cL)>0$.

According to the technical theory of pretwisted beams, the average strain on every cross section is constant along the yarn [see Eq. (41)], i.e.,

$$\bar{\varepsilon}_{zz}=\frac{\bar{N}}{EA}=\varepsilon_N \tag{66}$$

On the other hand, the homogenization scheme of section "Analogy of the One-Dimensional Dipolar Gradient 221 Elasticity Model" predicts an axial strain distribution of the form

$$\varepsilon(z)=\frac{dw_1(z)}{dz}=\frac{\bar{N}}{EA}+\frac{\bar{P}-\bar{N}}{EA}H\left(\frac{z}{L},\frac{\ell}{cL}\right) \tag{67}$$

where

$$\frac{\bar{P}-\bar{N}}{EA}=\frac{1}{c^2(\alpha_0)}\frac{S}{A}\frac{1}{GJ}\left(\frac{\alpha_0^2S}{A}\bar{N}-\alpha_0\bar{T}\right) \tag{68}$$

According to Eq. (67), the strain distribution along the yarn differs from the classical strain $\bar{N}/EA=\varepsilon_N$, and the difference depends on the sign of $\bar{P}-\bar{N}$. In particular, the axial strains are higher in a pretwisted beam when (see also Fig. 4)

$$\bar{P}>\bar{N}\quad\text{or}\quad\frac{\alpha_0^2S}{A}\bar{N}>\alpha_0\,\bar{T} \tag{69}$$

In the case of uniaxial tension ($\bar{N}>0$, $\bar{T}=0$), the inequalities [Eq. (69)] are satisfied and the axial strains are predicted to be higher than ε_N in a pretwisted yarn. This implies, in turn, that, as $\bar{N}$ increases, the axial strain will reach the intrinsic strength earlier in a pretwisted yarn. This is in accord with experimental observations (Hearle et al. 1969).

In the more general case where the loading includes both an axial load $\bar{N}$ and a torque $\bar{T}$, the behavior of the yarn depends on the sign of $\bar{P}-\bar{N}$ or, equivalently, on the sign of $(\alpha_0^2S/A)\bar{N}-\alpha_0\bar{T}$, as depicted in Eq. (69).

Applications

This section will examine the classical problem of tension–torsion coupling with uniform twist ($d\phi/dz=\text{const.}$) and present closed-form solutions for cross sections of various shapes, including the special case of thin-walled open cross sections.

In the case of uniform twist, Eqs. (19) and (20) define $d\phi/dz$ and dw_1/dz as follows:

$$\frac{d\phi(z)}{dz} = \frac{1}{GJc^2}\left[T(z) - \frac{\alpha_0 S}{A}N(z)\right] \quad \text{and}$$
$$\frac{dw_1(z)}{dz} = \frac{1}{c^2}\left[\left(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G}\right)\frac{N(z)}{EA} - \frac{\alpha_0 S}{A}\frac{T(z)}{GJ}\right] \tag{70}$$

so that

$$\frac{d\phi/dz}{dw/dz} = -\frac{\frac{\alpha_0 S}{A}N(z) - T(z)}{GJ(1 + \frac{\alpha_0^2 K}{J}\frac{E}{G})\frac{N(z)}{EA} - \frac{\alpha_0 S}{A}T(z)} \tag{71}$$

The last equation shows that, in the case of uniform twist,

$$T = 0 \Rightarrow \frac{d\phi}{dw_1} = -\frac{\frac{\alpha_0 S}{J}}{\frac{G}{E} + \frac{\alpha_0^2 K}{J}} < 0 \tag{72}$$

and

$$N = 0 \Rightarrow \frac{d\phi}{dw_1} = -\frac{A}{\alpha_0 S} < 0 \tag{73}$$

The negative values on the right-hand-sides of Eqs. (72) and (73) verify the well-known result that tension–torsion coupling of a free-to-warp pretwisted beam is negative, i.e., the beam untwists in extension and contracts when a torque in the direction of the pretwist is applied (see also Biot 1939). Note that in all of the aforementioned results, one can model approximately transversely anisotropic fibers with longitudinal modulus E and shear modulus G, as suggested by Hodges (1980).

Solid Cross Sections

Elliptical Cross Section

An elliptical cross section with semi-axes a and $b < a$ is considered (Fig. 5).

In this case (Sokolnikoff 1956)

$$\Psi = -\frac{a^2 - b^2}{a^2 + b^2}\eta\zeta, \quad J = \frac{\pi a^3 b^3}{a^2 + b^2}, \quad J_\omega = \frac{\pi a^3 b^3}{24}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2$$
$$I_p = \pi ab(a^2 + b^2)/4 \tag{74}$$

$$K = \frac{\pi ab}{24}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 (3a^4 - 2a^2b^2 + 3b^4),$$
$$R = 0, \quad S = \frac{\pi ab}{4}\frac{(a^2 - b^2)^2}{a^2 + b^2} \tag{75}$$

$$\ell = \frac{\sqrt{6}}{12}\sqrt{\frac{E}{G}}\frac{a^2 - b^2}{\sqrt{a^2 + b^2}},$$
$$c = \sqrt{1 + \frac{\alpha_0^2}{48}\frac{E}{G}\left(\frac{a^2 - b^2}{a\,b}\right)^2 \frac{3\,a^4 + 2\,a^2\,b^2 + 3\,b^4}{a^2 + b^2}} \tag{76}$$

In this example and the others to follow, the isotropic relation $E/G = 2(1 + v)$ has been used, where v = Poisson's ratio. In the case of uniaxial tension ($\bar{N} > 0$, $\bar{T} = 0$) and for uniform twist, Eq. (72) implies that

$$a\frac{d\phi}{dw_1} = -\frac{6(a\,\alpha_0)}{\frac{12}{1+\nu}\frac{\rho}{(1-\rho)^2} + \frac{3-2\rho+3\rho^2}{1+\rho}(a\alpha_0)^2}, \qquad \rho = \left(\frac{b}{a}\right)^2 \tag{77}$$

The quantity $\alpha_0 a$ above is the spiral angle θ_0^{spiral}, i.e., the angle formed between the longitudinal fibers of the beam at the major axes ($x = \pm a$, $y = 0$) of the ellipse and the beam axis, due to pretwist. For example, $a\alpha_0 = 1$ corresponds to a spiral angle of approximately 57.3°. Fig. 6 shows the variation of the tension–torsion coupling $ad\phi/dw_1$ with the spiral angle due to pretwist $a\alpha_0$, for a pretwisted beam of elliptical cross section with $b/a = 0.1$, $\nu = 0.3$, and for the case of uniform twist.

On the same plot the numerical results of Kosmatka (1992) are shown; Kosmatka solved the problem by using a finite-element scheme. It is interesting that the analytical solution [Eq. (77)] agrees very well with the numerical solution, even for large values of pretwist. In his numerical formulation, Kosmatka (1992) used a displacement field that accounted for possible deformations on the plane of the cross section and determined the warping function Ψ independently, i.e., without the assumption that Ψ is the Saint-Venant warping function of a similar beam without pretwist. The agreement of the present analytical solution with Kosmatka's (1992) solution suggests that the deformation of the cross section on its plane is indeed insignificant, and the Saint-Venant warping function can be used even for large pretwist $\alpha_0 a$.

Equilateral Triangle

Consider the case of a cross section with a shape of an equilateral triangle of height a, as shown in Fig. 7.

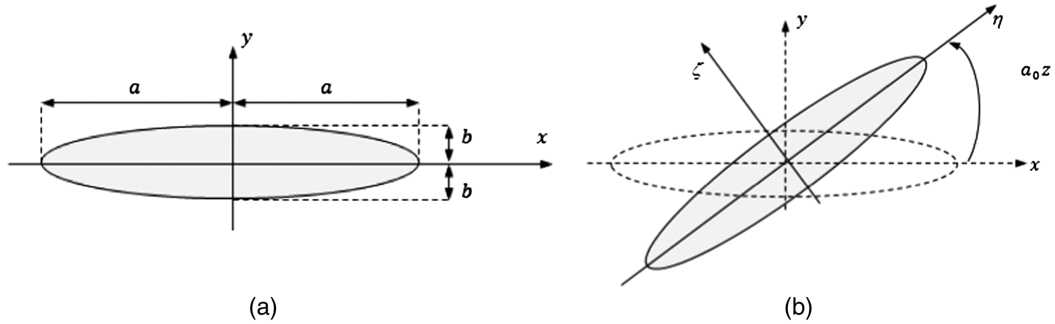

Fig. 5. Elliptical cross section of the fiber: (a) with no initial twist; (b) with initial twist

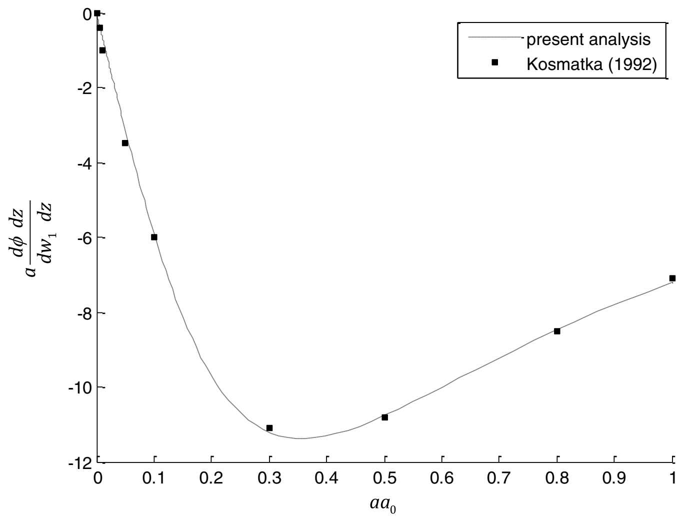

Fig. 6. Variation of the normalized rotational displacement increment $ad\phi$ to axial displacement increment dw_1 with the spiral angle due to pretwist aa_0, when an axial force is applied to a pretwisted beam of elliptical cross section with $b/a = 0.1$, $\nu = 0.3$, and $\ell = 0$

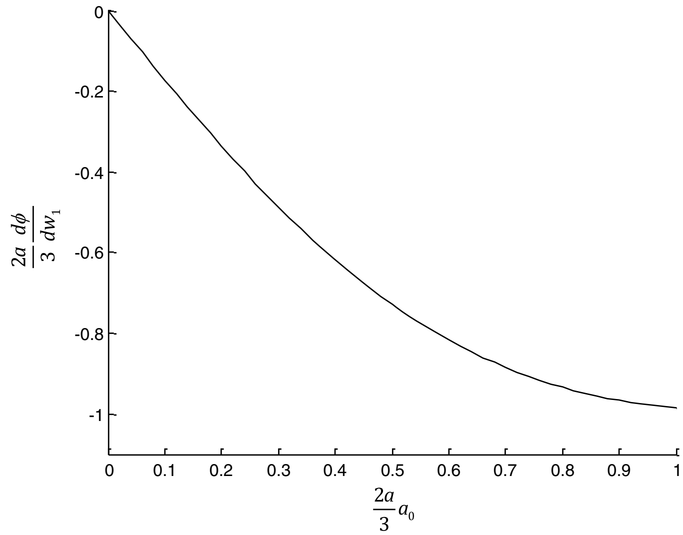

Fig. 8. Variation of $(2a/3)/(d\phi/dw_1)$ with $(2a/3)a_0$ for a cross section with a shape of an equilateral triangle in the case of uniaxial tension ($\bar{N} > 0, \bar{T} = 0$) for $\ell = 0$ and $\nu = 0.3$

In this case

$$\Psi = -\frac{1}{2\,a}(\zeta^3 - 3\eta^2\zeta), \qquad J = \frac{\sqrt{3}}{45}a^4, \qquad J_\omega = \frac{\sqrt{3}}{1701}a^6 \quad (78)$$

$$K = \frac{7\sqrt{3}}{2430}a^6, \qquad R = 0, \qquad S = \frac{2\sqrt{3}}{135}a^4 \quad (79)$$

$$\ell = \frac{\sqrt{42}}{126}\sqrt{\frac{E}{G}}a, \qquad c = \sqrt{1 + \frac{1}{10}(\alpha_0 a)^2\frac{E}{G}} \quad (80)$$

In the case of uniaxial tension ($\bar{N} > 0$, $\bar{T} = 0$), and for uniform twist, Eq. (72) implies that

$$\frac{2a}{3}\frac{d\phi}{dw_1} = -\frac{16 \times (\frac{2a}{3}\alpha_0)}{\frac{12}{1+\nu} + 7 \times (\frac{2a}{3}\alpha_0)^2} \quad (81)$$

The quantity $\alpha_0(2a/3)$ above is the spiral angle θ_0^{spiral}, i.e., the angle formed between the longitudinal fibers of the beam at the apex $[x = (2a/3), y = 0]$ and the beam axis, due to pretwist. Fig. 8 shows the variation of $(2a/3)(d\phi/dw_1)$ with $\alpha_0(2a/3)$ for the case of uniaxial tension ($\bar{N} > 0$, $\bar{T} = 0$) for uniform twist and $\nu = 0.3$.

Open Thin-Walled Cross Sections

Consider the special case of a uniform cylindrical beam with an open thin-walled cross section that is constant along the beam. For such cross sections, the evaluation of quantities such as K,

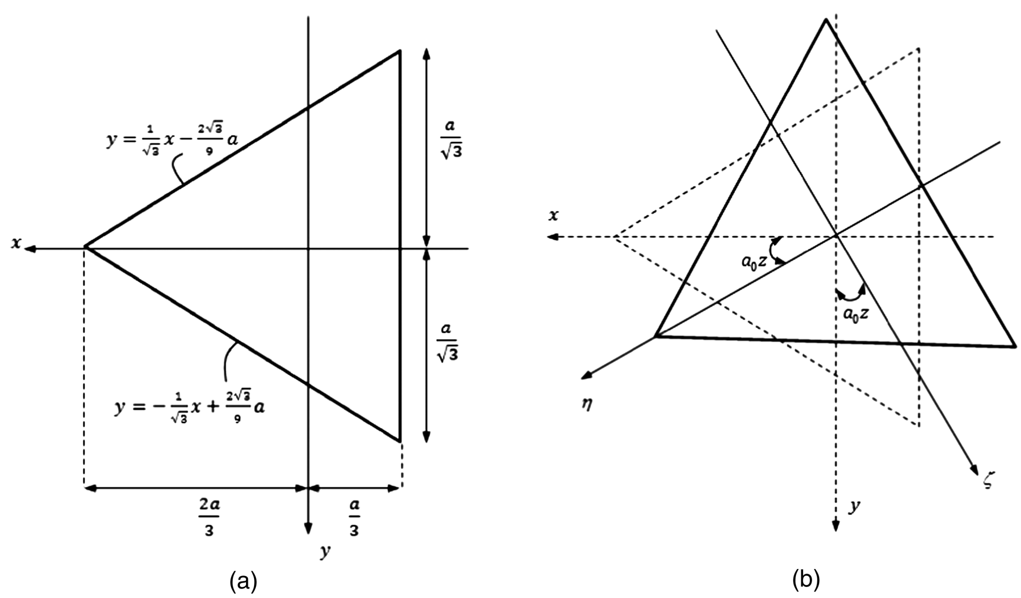

Fig. 7. Equilateral triangle cross section: (a) with no initial twist; (b) with initial twist

R, and S introduced in section "Tension and Torsion of Pretwisted Beams" requires consideration of the so-called secondary warping of the cross section, as discussed in the following.

On each cross section, the arc length s is measured along the middle line C in the counterclockwise direction, starting from one end of the section, where $s = 0$. At each point on the middle line C, a normal direction n is introduced in such a way that the system $n - s - z$ is right-handed, with $n = 0$ on C and $-[t(s)/2] \leq n \leq [t(s)/2]$, where $t(s)$ = thickness of the cross section normal to the middle line. The beam is thin-walled in the sense that the maximum value of $t(s)$ along C is small compared with the dimensions of the cross section.

Wagner (1936) showed that the warping of a thin-walled cross section (with no pretwist) can be written in the form [see also Goodier (1962, pp. 36–19); Librescu and Song (2006, p. 18)]

$$w(n,s,z) = w_1(z) + \frac{d\phi(z)}{dz}\Psi(n,s),$$
$$\Psi(n,s) = -\omega^*(s) + r_s(s)n \tag{82}$$

where $\omega^*(s)$ = so-called principal sectorial area of the middle line (Vlasov 1961); $r_s(s) = \mathbf{r}(s)\cdot\mathbf{p}(s)$ = distance between the shear center and the normal to the middle line at s; $\mathbf{r}(s)$ = position vector of a point on the cross section; and $\mathbf{p}(s)$ = unit vector tangent to the middle line and in the direction of increasing s. The term $-[d\phi(z)/dz]\omega^*(s)$ in Eq. (82) defines the out-of-plane displacement of the points on the middle line ($n = 0$), and the term $[d\phi(z)]/dz r_s(s)n$ is the secondary warping of the points off the middle line, where $n \neq 0$. The principal sectorial area is defined so that $\int_C \omega^*(s)ds = 0$.

The torsional constant J and the sectorial moment J_ω can be written in the form

$$J = \frac{1}{3}\int_C t^3(s)ds \tag{83}$$

and

$$J_\omega = \int_A \Psi^2 dA = \int_C [\omega^{*2}(s)t(s) + \frac{1}{12}r_s^2(s)t^3(s)]ds \tag{84}$$

When the cross section is such that $\omega^*(s) \neq 0$ and has constant thickness t, the material length ℓ is inversely proportional to t:

$$\ell = \sqrt{\frac{E}{G}\frac{J_\omega}{J}} = \sqrt{\frac{E}{G}\frac{H^2}{t}}, \qquad H^2 \cong \sqrt{\frac{3}{L_C}\int_C \omega^{*2}ds} \tag{85}$$

where $L_C = \int_C ds$ = length of the middle line $[(t/L_C) << 1]$; H = dimensions of length; and the contribution of secondary warping has been ignored.

In the special case where $\omega^*(s) = 0$ (e.g., for a thin-walled rectangular cross section), ℓ is independent of the thickness t:

$$\ell = \sqrt{\frac{E}{G}\frac{J_\omega}{J}} = \sqrt{\frac{E}{G}}B, \qquad B \cong \frac{1}{2}\sqrt{\frac{1}{L_C}\int_C r_s^2 ds} \tag{86}$$

Thin Rectangular Cross Section

Consider first a thin rectangular cross section with dimensions $b \times t$, where $t/b << 1$ (Fig. 9).

In this case, $\omega^* = 0$ and the warping is secondary so that

$$\Psi \cong -\eta\zeta, \qquad J \cong \frac{1}{3}bt^3, \qquad J_\omega \cong \frac{1}{144}b^3t^3 \tag{87}$$

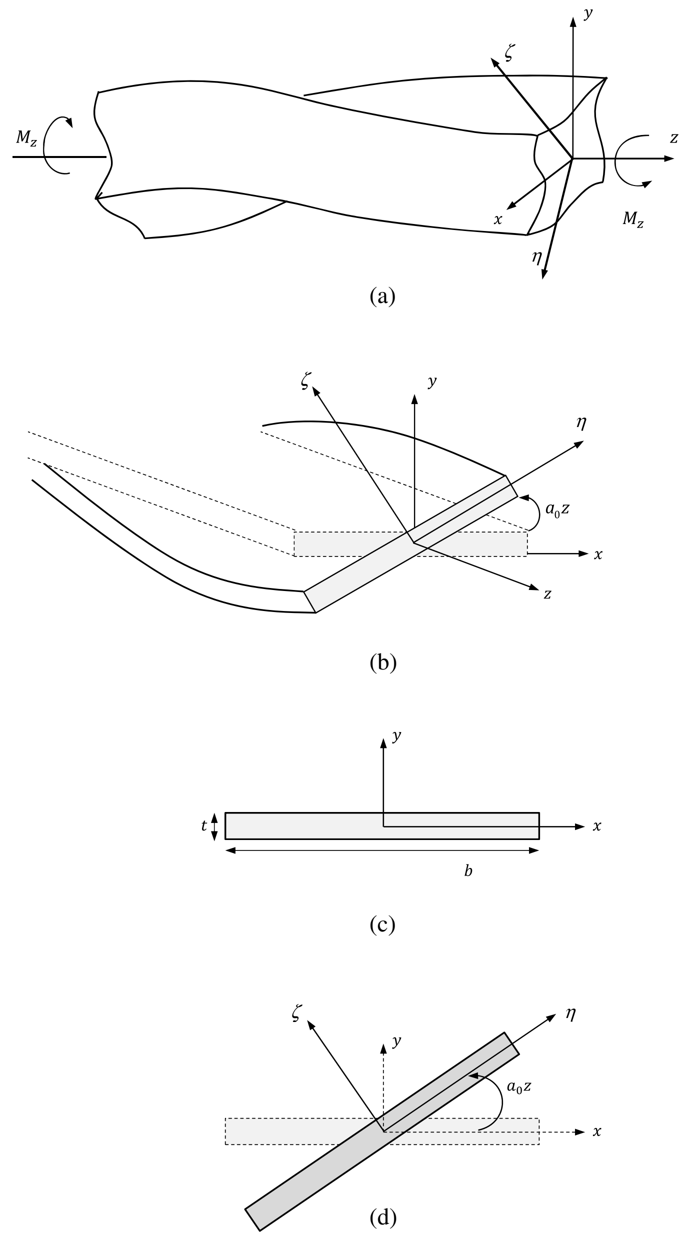

Fig. 9. Stereoscopic view of the thin-wall cross section of the fiber: (a) with initial twist; (b) with no initial twist; front view of the cross section: (c) with no initial twist; (d) with initial twist

$$K \cong \frac{1}{80}tb^5, \qquad R = 0, \qquad S \cong \frac{1}{12}tb^3 \tag{88}$$

$$\ell \cong \frac{\sqrt{3}}{12}\sqrt{\frac{E}{G}}b, \qquad c \cong \sqrt{1 + \frac{1}{15}\left(\alpha_0\frac{b}{2}\right)^2\left(\frac{b}{t}\right)^2\frac{E}{G}} \tag{89}$$

Note that the material length is proportional to the width b of the cross section and independent of the thickness t.

In the case of uniaxial tension ($\bar{N} > 0$, $\bar{T} = 0$) and for uniform twist, Eq. (72) implies that

$$\frac{b}{2}\frac{d\phi}{dw_1} = -\frac{15(\frac{b}{2}\alpha_0)}{\frac{30}{1+\nu}\frac{\delta}{1-\delta} + \frac{9-10\delta+9\delta^2}{1-\delta}(\frac{b}{2}\alpha_0)^2}, \qquad \delta = \left(\frac{t}{b}\right)^2 \tag{90}$$

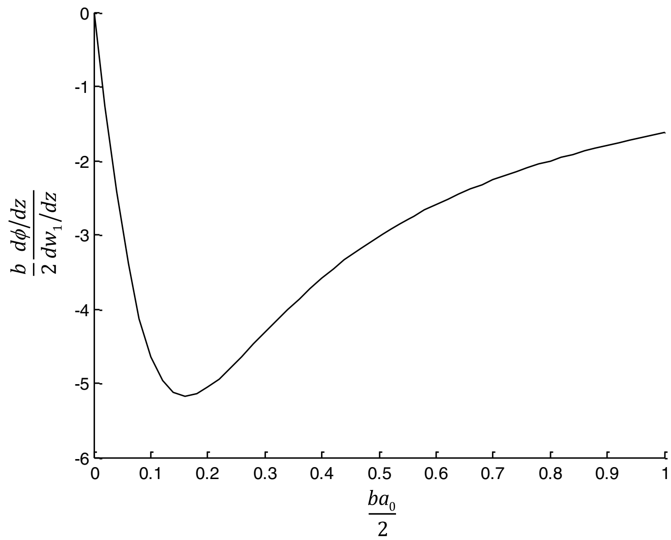

Fig. 10. Variation of $(b/2)[(d\phi/dz)/(dw_1/dz)]$ with $(b/2)a_0$ for beam of a thin rectangular cross section in the case of uniaxial tension $(\bar{N} > 0, \bar{T} = 0)$ for $\ell = 0$, $t/b = 1/10$, and $\nu = 0.3$

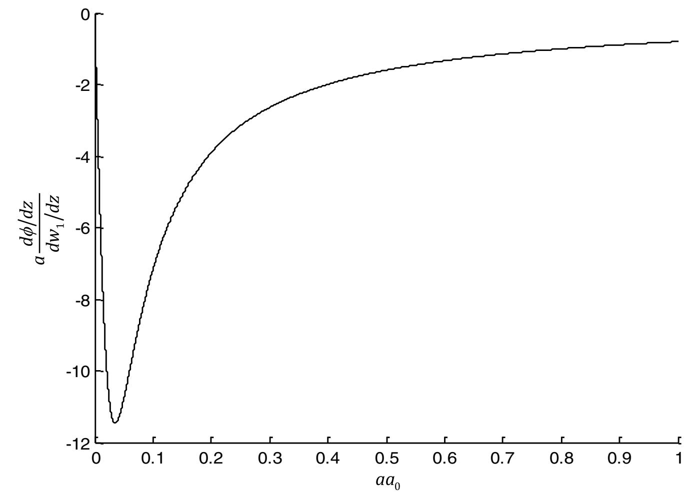

Fig. 12. Variation of $a(d\phi/dw_1)$ with aa_0 for the Z-shaped cross section in the case of uniaxial tension $(\bar{N} > 0,\ \bar{T} = 0)$ for $\ell = 0$, $t/b = 1/10$, and $\nu = 0.3$

The quantity $\alpha_0(b/2)$ above is the spiral angle θ_0^{spiral}, i.e., the angle formed between the longitudinal fibers of the untwisted beam at $(x = \pm b/2,\ y = 0)$ and the beam axis, due to pretwist. Fig. 10 shows the variation of $(b/2)(d\phi/dw_1)$ with $a_0(b/2)$ for $t/b = 0.1$ and $\nu = 0.3$ for the case of uniform twist.

Thin-Walled Z-Section

Finally, consider the case of the thin-walled Z-section shown in Fig. 11 with $t/a << 1$.

In this case the warping function has the form

$$\Psi = \begin{cases} -a\left(\frac{a}{4}+\eta\right) - \underline{\underline{\eta(\zeta+a)}} & \text{along AB} \\ -\frac{a^2}{4} + \underline{\underline{\zeta\,\eta}} & \text{along BC} \\ -a\left(\frac{a}{4}-\eta\right) - \underline{\underline{\eta(\zeta-a)}} & \text{along CD} \end{cases} \tag{91}$$

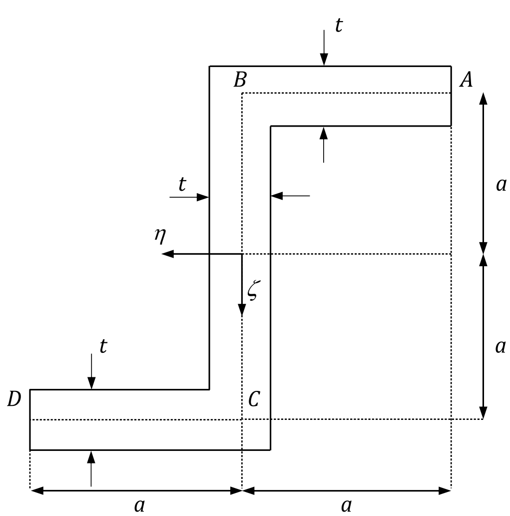

Fig. 11. Thin-walled Z-section

The underlined terms in the above expression for Ψ correspond to secondary warping and account for the variation of Ψ in the thickness direction. The contribution of these terms to the value of Ψ is insignificant; however, these terms affect significantly the derivatives of Ψ that are required for the evaluation of K, R, and S. Also, since the cross section does not have an axis of symmetry, parameter R and the surface length ℓ_1 do not vanish. In this case, the following is true:

$$J \cong \frac{4}{3}t^3 a, \qquad J_\omega \cong \frac{5}{12}ta^5 \tag{92}$$

$$K \cong \frac{62}{15}ta^5, \quad R \cong \frac{2}{3}ta^5, \quad S \cong \frac{10}{3}ta^3 \tag{93}$$

$$\ell \cong \frac{\sqrt{5}}{4}\sqrt{\frac{E}{G}}\frac{a^2}{t}, \qquad c \cong \sqrt{1 + \frac{61}{60}(\alpha_0 a)^2\left(\frac{a}{t}\right)^2\frac{E}{G}} \tag{94}$$

Note that the material length ℓ is now inversely proportional to the thickness t.

In the case of uniaxial tension $(\bar{N} > 0,\ \bar{T} = 0)$ and for uniform twist, Eq. (72) implies that

$$a\frac{d\phi/dz}{dw_1/dz} = -\frac{5(a\,\alpha_0)}{\frac{1}{1+\nu}\left(\frac{t}{a}\right)^2 + \frac{31}{5}(a\alpha_0)^2} \tag{95}$$

The quantity $\alpha_0 a$ above is the spiral angle θ_0^{spiral}, i.e., the angle formed between the longitudinal fibers of the beam at the corners $(x = 0,\ y = \pm a)$ and the beam axis, due to pretwist. Fig. 12 shows the variation of $a(d\phi/dw_1)$ with $\alpha_0 a$ for the case of uniaxial tension $(\bar{N} > 0,\ \bar{T} = 0)$ for uniform twist, $t/b = 0.1$, and $\nu = 0.3$.

Conclusions

An interesting analogy between the technical theory of a pretwisted beam loaded by tension and torsion and a one-dimensional dipolar-gradient elasticity model is presented. The microstructural lengths g and m required by the gradient theory can be quantified directly from the geometric properties of the cross section, the amount of pretwist, and the elastic properties of the beam. A high value of E/G could attenuate size phenomena by enhancing the internal length g. The generalized loads of the gradient elasticity are also well explained in terms of classical axial loads and torsional moments.

The classical solution of the technical theory provides the microstresses that are implied by the gradient elasticity theory, which acts as a homogenization theory that averages the details of the classical solution. The average tensile microstrain appears to be controlled by the classical axial stress; therefore, any damage evolution calculations should be based on the Cauchy stresses of the classical solution rather than the total stresses of the gradient theory, which includes the coupled stresses as well.

The surface type of length m that enters the boundary conditions of the gradient theory is attributed to the lack of symmetry of the cross section; cross sections with at least one axis of symmetry have zero surface length $(m = 0)$.

The results presented herein provide a novel micromechanical approach for tensile yarns that are manufactured as pretwisted fibers. The advantage of the present formulation is that one can formulate a system of beams, such as the presented ones, in a very compact way, constructing two- or three-dimensional trusses or other patterns of woven textiles. This approach is also applicable to smart textiles, which are magnetostrictive fibers that are pretwisted deliberately by external electromagnetic fields, taking advantage of the Wiedemann effect [e.g., see Malyugin (1991); Wajchman et al. (2008)].

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