J Elast (2011) 104:45–70
DOI 10.1007/s10659-011-9308-7

Plane-Strain Problems for a Class of Gradient Elasticity Models—A Stress Function Approach

Nikolaos Aravas

Received: 11 October 2010 / Published online: 26 January 2011

Abstract The plane strain problem is analyzed in detail for a class of isotropic, compressible, linearly elastic materials with a strain energy density function that depends on both the strain tensor $\boldsymbol{\varepsilon}$ and its spatial gradient $\nabla\boldsymbol{\varepsilon}$. The appropriate Airy stress-functions and double-stress-functions are identified and the corresponding boundary value problem is formulated. The problem of an annulus loaded by an internal and an external pressure is solved.

Keywords Gradient elasticity · Stress functions

Mathematics Subject Classification (2000) 74A35

1 Introduction

Theories with intrinsic- or material-length-scales find applications in the modeling of size-dependent phenomena. In elasticity, length scales enter the constitutive equations through the elastic strain energy function, which in this case depends not only on the strain tensor but also on gradients of the rotation and strain tensors; in such cases we refer to "gradient elasticity" theories.

A first attempt to incorporate length scale effects in elasticity was made by Mindlin [34], Koiter [27, 28] and Toupin [42]. They solved also a number of problems and demonstrated the effects of the material length scales that enter the strain-gradient elasticity theories (Mindlin and Tiersten [37], Mindlin [34, 35], Koiter [28]). Several theoretical issues related to strain-gradient elasticity were addressed later by Germain [23–25]. More recently,

The paper is dedicated to the memory of Professor Donald Carlson.

N. Aravas (✉)
Department of Mechanical Engineering, University of Thessaly, Volos 38334, Greece
e-mail: aravas@uth.gr

N. Aravas
The Mechatronics Institute, Center for Research and Technology—Thessaly (CERETETH),
1st Industrial Area, Volos 38500, Greece

a variety of "non-local" or "gradient-type" theories have been used in order to introduce material length scales into constitutive models (Aifantis [1], Pijaudier-Cabot and Bazant [39], Vardoulakis *et al.* [44–46], de Borst [9], Fleck *et al.* [16–18], Leblond *et al.* [32], Tvergaard and Needleman [43]). A common feature of all the aforementioned theories is the non-symmetry of the true stress tensor, and the existence of couple and higher order stresses. Some of the early developments in the theory are summarized in the Nowacki's book [38] that was published in the 80s. Lazar and Maugin [29–31] have solved a series of problems including dislocations, disclinations, and line forces in the context of gradient elasticity. A summary of the applicability of gradient elasticity to certain micro/nano problems has been given recently by Aifantis [3].

In the present work, the plane strain problem is analyzed in detail for a class of isotropic, linearly elastic materials with a strain energy density function that depends on both the strain tensor $\boldsymbol{\varepsilon}$ and its spatial gradient $\nabla\boldsymbol{\varepsilon}$. The appropriate Airy stress-functions and double-stress-functions are identified and the corresponding boundary value problem is formulated. The problem of an annulus loaded by an internal and an external pressure is solved and the effects of strain gradients are examined. The special cases of a thin-walled tube and an infinite body with a cylindrical hole are also analyzed.

Standard notation is used throughout. Boldface symbols denote tensors the orders of which are indicated by the context. The usual summation convention is used for repeated Latin and Greek indices of tensor components with respect to a fixed Cartesian coordinate system with base vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$. Latin indices take the values $(1, 2, 3)$, whereas Greek indices range over the integers $(1, 2)$, unless indicated otherwise. A comma followed by a subscript, say i, denotes partial differentiation with respect to the spatial coordinate x_i, i.e., $f_{,i} = \partial f/\partial x_i$. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be vectors, $\mathbf{A}$, $\mathbf{B}$ second-order tensors, and $\boldsymbol{\kappa}$, $\boldsymbol{\mu}$ third-order tensors; the following products are used in the text: $(\mathbf{ab})_{ij} = a_i b_j$, $(\mathbf{abc})_{ijk} = a_i b_j c_k$, $(\mathbf{aA})_{ijk} = a_i A_{jk}$, $(\mathbf{a}\cdot\mathbf{A})_i = a_j A_{ji}$, $(\mathbf{aA})_{ijk} = a_i A_{jk}$, $\mathbf{A} : \mathbf{B} = A_{ij} B_{ij}$, $(\mathbf{a}\cdot\boldsymbol{\mu})_{ij} = a_k \mu_{kij}$, $\boldsymbol{\kappa} : \boldsymbol{\mu} = \kappa_{ijk}\mu_{ijk}$, $(\nabla\cdot\boldsymbol{\mu})_{ij} = \mu_{kij,k}$,[1] $(\boldsymbol{\mu}\cdot\nabla)_{ij} = \mu_{ijk,k}$, $(\mathbf{a}\times\mathbf{A})_i = e_{ipq} a_p A_{qj}$, $(\nabla\times\mathbf{A})_{ij} = e_{ipq} A_{qj,p}$, $(\mathbf{A}\times\nabla)_{ij} = -e_{jpq} A_{iq,p}$, and $(\nabla\times\boldsymbol{\kappa})_{ijk} = e_{ipq}\kappa_{qjk,p}$, where e_{ijk} is the alternating symbol.

2 The Constitutive Model

We consider the class of elastic materials in which the elastic strain energy density function W depends on the infinitesimal strain tensor and its spatial gradient. Mindlin and Eshel [36] have shown that for an isotropic, compressible, linearly elastic material, the general form of W is

$$W(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = G\left(\varepsilon_{ij}\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ii}\varepsilon_{jj}\right) + a_1\kappa_{iik}\kappa_{kjj} + a_2\kappa_{ijj}\kappa_{ikk} + a_3\kappa_{iik}\kappa_{jjk} + a_4\kappa_{ijk}\kappa_{ijk} + a_5\kappa_{ijk}\kappa_{kji}, \tag{1}$$

where $\boldsymbol{\varepsilon} = \frac{1}{2}(\mathbf{u}\nabla + \nabla\mathbf{u})$ is the infinitesimal strain tensor, $\mathbf{u}$ the displacement field, $\boldsymbol{\kappa} = \nabla\boldsymbol{\varepsilon}$ $(\kappa_{ijk} = \kappa_{ikj} = \varepsilon_{jk,i})$ the strain gradient third-order tensor, G the elastic shear modulus, ν Poisson's ratio, and $(a_1, a_2, a_3, a_4, a_5)$ material constants.

[1] With respect to a Cartesian system x_i with unit base vectors $\mathbf{e}_i$, the gradient operator is written in the form $\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i} = \mathbf{e}_i \partial_i$.

We consider the special case in which

$$a_1 = a_3 = a_5 = 0, \qquad a_2 = \frac{\nu}{1-2\nu} G\ell^2, \qquad a_4 = G\ell^2, \tag{2}$$

where ℓ is a material length, so that

$$W(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}) = G\left[\varepsilon_{ij}\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ii}\varepsilon_{jj} + \ell^2\left(\kappa_{ijk}\kappa_{ijk} + \frac{\nu}{1-2\nu}\kappa_{ijj}\kappa_{ikk}\right)\right]. \tag{3}$$

We define the stress-like and double-stress-like quantities $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ as follows:

$$\tau_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} = 2G\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right), \tag{4}$$

and

$$\mu_{ijk} = \frac{\partial W}{\partial \kappa_{ijk}} = 2G\ell^2\left(\kappa_{ijk} + \frac{\nu}{1-2\nu}\kappa_{ipp}\delta_{jk}\right). \tag{5}$$

The above equation can be inverted to yield

$$\varepsilon_{ij} = \frac{1}{2G}\left(\tau_{ij} - \frac{\nu}{1+\nu}\tau_{kk}\delta_{ij}\right) \quad \text{and} \quad \kappa_{ijk} = \frac{1}{2G\ell^2}\left(\mu_{ijk} - \frac{\nu}{1+\nu}\mu_{ipp}\delta_{jk}\right). \tag{6}$$

Equations (5) imply also that $\mu_{ijk} = \ell^2\tau_{jk,i}$ and $\mu_{ijk,i} = \ell^2\tau_{jk,ii} = \ell^2\nabla^2\tau_{jk}$, or

$$\boldsymbol{\mu} = \ell^2\nabla\boldsymbol{\tau} \quad \text{and} \quad \nabla\cdot\boldsymbol{\mu} = \ell^2\nabla^2\boldsymbol{\tau}. \tag{7}$$

The quantity $\boldsymbol{\tau} - \nabla\cdot\boldsymbol{\mu}$ that enters (20) below can be written as

$$\boldsymbol{\tau} - \nabla\cdot\boldsymbol{\mu} = \lambda\varepsilon_{kk}\boldsymbol{\delta} + 2\mu\boldsymbol{\varepsilon} - \ell^2\nabla^2\left(\lambda\varepsilon_{kk}\boldsymbol{\delta} + 2\mu\boldsymbol{\varepsilon}\right). \tag{8}$$

The right hand side of the above equation is formally similar to the expression used for the stress tensor by Aifantis [2] and Altan and Aifantis [4] in their version of a gradient elasticity theory.

The strain energy density function given in (3) can be written also in the form

$$W = \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\mu}:\boldsymbol{\kappa} = \frac{1}{2}\boldsymbol{\tau}:\boldsymbol{\varepsilon} + \frac{\ell^2}{2}\nabla\boldsymbol{\tau}:\nabla\boldsymbol{\varepsilon}. \tag{9}$$

3 The Compatibility Conditions

The compatibility conditions for the strain tensor have the well know form

$$\mathbf{S} \equiv \nabla\times\boldsymbol{\varepsilon}\times\nabla = -e_{ipm}e_{jqn}\varepsilon_{pq,mn}\mathbf{e}_i\mathbf{e}_j = \mathbf{0}. \tag{10}$$

The above equations are the necessary and sufficient conditions for the existence of a displacement field $\mathbf{u}(\mathbf{x})$ such that the relation $\boldsymbol{\varepsilon} = (1/2)(\mathbf{u}\nabla + \nabla\mathbf{u})$ is satisfied. If the region of interest $\mathcal{B}$ is simply connected, then the ε-compatibility conditions (10) guarantee also that $\mathbf{u}(\mathbf{x})$ is single valued. An interesting discussion of the strain compatibility conditions and their relationship to a Stokes theorem for second-order tensor fields has been given by Fosdick and Royer-Carfagni [19].

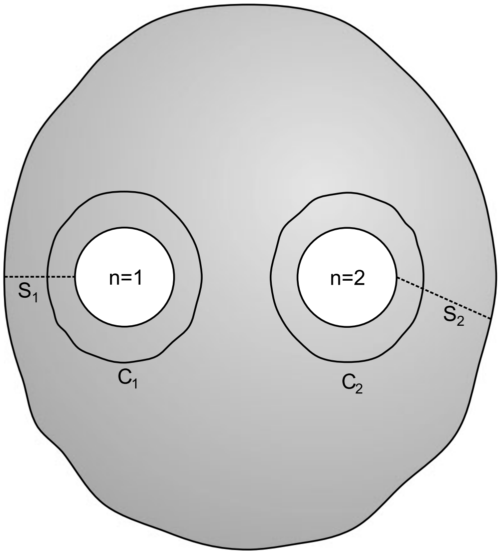


Fig. 1 Triply-connected region ($N = 2$)

The definition of the strain gradient $\boldsymbol{\kappa} = \nabla\boldsymbol{\varepsilon}$ implies the κ-compatibility conditions:

$$\mathbf{P} \equiv \nabla \times \boldsymbol{\kappa} = e_{ipq}\kappa_{qjk,p}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \mathbf{0}. \tag{11}$$

Conversely, satisfaction of the κ-compatibility conditions implies the existence of a tensor field $\boldsymbol{\varepsilon}(\mathbf{x})$ such that $\boldsymbol{\kappa} = \nabla\boldsymbol{\varepsilon}$. Furthermore, if the region is simply connected, $\boldsymbol{\varepsilon}(\mathbf{x})$ is single-valued (Courant and John [14]).

If $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are related to $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ though (6), the above compatibility conditions can be written in the form

$$\begin{aligned}\mathbf{S} &= \frac{1}{2G}\left\{\nabla \times \boldsymbol{\tau} \times \nabla + \frac{\nu}{1+\nu}\left[\nabla^2\tau_{kk}\boldsymbol{\delta} - \nabla\left(\nabla\tau_{kk}\right)\right]\right\} \\ &= \frac{1}{2G}\left[-e_{ipm}e_{jqn}\tau_{pq,mn} + \frac{\nu}{1+\nu}\left(\nabla^2\tau_{kk}\delta_{ij} - \tau_{kk,ij}\right)\right]\mathbf{e}_i\mathbf{e}_j = \mathbf{0},\end{aligned} \tag{12}$$

and

$$\begin{aligned}\mathbf{P} &= \frac{1}{2G\ell^2}\left[\nabla \times \boldsymbol{\mu} - \frac{\nu}{1+\nu}\left(\nabla \times \boldsymbol{\mu}\right)_{irr}\mathbf{e}_i\boldsymbol{\delta}\right] \\ &= \frac{e_{ipq}}{2G\ell^2}\left(\mu_{qjk,p} - \frac{\nu}{1+\nu}\mu_{qrr,p}\delta_{jk}\right)\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k = \mathbf{0}.\end{aligned} \tag{13}$$

We conclude this section with a discussion of the compatibility conditions in multiply connected domains. Assume that the region $\mathcal{B}$ is $(N+1)$-tuply connected, i.e., it has N "holes". Starting with $\mathcal{B}$, we can form a simply-connected domain by inserting N interior surfaces S_n $(n = 1, 2, \ldots, N)$ between the outside boundary and each hole (doted lines in Fig. 1).

For $\mathbf{u}(\mathbf{x})$ to be single valued, in addition to the ε-compatibility conditions (10), the following $6N$ conditions are required (e.g., see Boley and Weiner [7] pp. 92–95 and 97–100, Fung [21] pp. 104–107, Fraeijs de Veubeke [20] pp. 90–97):

$$\mathbf{S}^u \equiv \oint_{C_n} [\boldsymbol{\varepsilon} + \mathbf{x} \times (\nabla \times \boldsymbol{\varepsilon})] \cdot d\mathbf{x} = \oint_{C_n} \left(\varepsilon_{ij} - x_k e_{rik} e_{rpq} \varepsilon_{pj,q}\right) dx_j \mathbf{e}_i$$
$$= \oint_{C_n} \left[\varepsilon_{ij} - x_k \left(\varepsilon_{ij,k} - \varepsilon_{kj,i}\right)\right] dx_j \mathbf{e}_i$$
$$= -\oint_{C_n} \left[x_k \left(\varepsilon_{ik,j} + \varepsilon_{ij,k} - \varepsilon_{kj,i}\right)\right] dx_j \mathbf{e}_i = \mathbf{0}, \quad n = 1, 2, \ldots, N, \tag{14}$$

$$\mathbf{S}^\omega \equiv \oint_{C_n} (\nabla \times \boldsymbol{\varepsilon}) \cdot d\mathbf{x} = \oint_{C_n} e_{ipq} \varepsilon_{pj,q}\, dx_j \mathbf{e}_i = \mathbf{0}, \quad n = 1, 2, \ldots, N, \tag{15}$$

where C_n $(n = 1, 2, \ldots, N)$ are simple closed curves in $\mathcal{B}$ that surround only a single hole or alternatively, each cut only a single surface S_n as introduced above (see Fig. 1).

Equations (15) guarantee that the infinitesimal rotation field is single valued and, under these conditions, (14) guarantee that the displacement filed $\mathbf{u}(\mathbf{x})$ is single valued.

Similarly, it can be shown readily that for $\boldsymbol{\varepsilon}(\mathbf{x})$ to be single valued, in addition to the κ-compatibility conditions (11), the following $6N$ conditions are required:

$$\mathbf{P}^\varepsilon \equiv \oint_{C_n} (d\mathbf{x} \cdot \boldsymbol{\kappa}) = \oint_{C_n} \kappa_{kij}\, dx_k \mathbf{e}_i \mathbf{e}_j = \mathbf{0}, \quad n = 1, 2, \ldots, N. \tag{16}$$

If $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ are related to $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ though (6), the compatibility conditions on C_n (14)–(16) can be written in the form

$$\mathbf{S}^u = \frac{1}{2G} \oint_{C_n} \left[\boldsymbol{\tau} - \frac{\nu}{1+\nu} \tau_{kk} \boldsymbol{\delta} + \mathbf{x} \times \left(\nabla \times \boldsymbol{\tau} + \frac{\nu}{1+\nu} e_{ijp} \tau_{kk,p} \mathbf{e}_i \mathbf{e}_j\right)\right] \cdot d\mathbf{x}, \tag{17}$$

$$\mathbf{s}^\omega = \frac{1}{2G} \oint_{C_n} \left[\mathbf{x} \times \left(\nabla \times \boldsymbol{\tau} + \frac{\nu}{1+\nu} e_{ijp} \tau_{kk,p} \mathbf{e}_i \mathbf{e}_j\right)\right] \cdot d\mathbf{x}, \tag{18}$$

$$\mathbf{P}^\varepsilon = \frac{1}{2G\ell^2} \oint_{C_n} d\mathbf{x} \cdot \left(\boldsymbol{\mu} - \frac{\nu}{1+\nu} \mu_{ipp} \mathbf{e}_i \boldsymbol{\delta}\right), \tag{19}$$

where $n = 1, 2, \ldots, N$ in all three equations above.

4 The Boundary Value Problem

Consider a homogeneous elastic body that occupies a region $\mathcal{B}$ in a fixed reference configuration and obeys the constitutive equations (4) and (5). The field equations in $\mathcal{B}$ are

$$\nabla \cdot (\boldsymbol{\tau} - \nabla \cdot \boldsymbol{\mu}) + \mathbf{b} = \mathbf{0}, \tag{20}$$

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\mathbf{u}\nabla + \nabla \mathbf{u}), \quad \boldsymbol{\kappa} = \nabla \boldsymbol{\varepsilon}, \tag{21}$$

$$\boldsymbol{\tau} = 2G\left(\boldsymbol{\varepsilon} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\boldsymbol{\delta}\right), \quad \boldsymbol{\mu} = 2G\ell^2\left(\boldsymbol{\kappa} + \frac{\nu}{1-2\nu}\kappa_{ipp}\mathbf{e}_i\boldsymbol{\delta}\right) = \ell^2\nabla\boldsymbol{\tau}, \tag{22}$$

where $\mathbf{b}$ is the body force per unit volume. Let $\partial\mathcal{B}$ be the smooth boundary of $\mathcal{B}$. The corresponding boundary conditions are (Mindlin [34], Mindlin and Eshel [36])

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{on } \partial\mathcal{B}_u, \tag{23}$$

$$\mathbf{n}\cdot(\boldsymbol{\tau} - \nabla\cdot\boldsymbol{\mu}) - \mathbf{D}\cdot(\mathbf{n}\cdot\boldsymbol{\mu}) + (\mathbf{D}\cdot\mathbf{n})\,\mathbf{n}\cdot(\mathbf{n}\cdot\boldsymbol{\mu}) = \mathbf{P} \quad \text{on } \partial\mathcal{B}_P, \tag{24}$$

$$D\mathbf{u} = \bar{\mathbf{v}} \quad \text{on } \partial\mathcal{B}_v, \tag{25}$$

$$\mathbf{n}\cdot(\mathbf{n}\cdot\boldsymbol{\mu}) = \mathbf{R} \quad \text{on } \partial\mathcal{B}_R, \tag{26}$$

where $(\bar{\mathbf{u}}, \bar{\mathbf{v}}, \mathbf{P}, \mathbf{R})$ are known functions, $D = \mathbf{n}\cdot\nabla = \frac{\partial}{\partial n}$ is the normal derivative to $\partial\mathcal{B}$, $\mathbf{D} = \nabla - \mathbf{n}D$ is the "surface gradient" on $\partial\mathcal{B}$, $\partial\mathcal{B}_u \cup \partial\mathcal{B}_P = \partial\mathcal{B}_v \cup \partial\mathcal{B}_R = \partial\mathcal{B}$, and $\partial\mathcal{B}_u \cap \partial\mathcal{B}_P = \partial\mathcal{B}_v \cap \partial\mathcal{B}_R = \varnothing$.

The boundary value problem defined by (20)–(26) determines $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\tau}$, and $\boldsymbol{\mu}$. Mindlin and Eshel [36] have shown that the conditions $G > 0$ and $-1 < \nu < 1/2$ are sufficient for a unique solution (see also Georgiadis *et al.* [22]).

If the boundary $\partial\mathcal{B}$ is not smooth, additional boundary conditions are required on the edges of $\partial\mathcal{B}$ [34, 36].

The quantities $\mathbf{P}$ and $\mathbf{R}$ in (24) and (26) are work conjugate to $\mathbf{u}$ and $D\mathbf{u}$, and are referred to as the "generalized" or "auxiliary" or "mathematical" forces and double-forces respectively. It should be emphasized that $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are defined as work-conjugate quantities to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$, i.e.,

$$\boldsymbol{\tau} = \frac{\partial W}{\partial\boldsymbol{\varepsilon}} \quad \text{and} \quad \boldsymbol{\mu} = \frac{\partial W}{\partial\boldsymbol{\kappa}}, \tag{27}$$

and then (20) and the boundary conditions (23)–(26) are derived from Hamilton's principle (Mindlin [34], Mindlin and Eshel [36]). The principles of conservation of linear and angular momentum introduce the true stress $\boldsymbol{\sigma}$ and the true couple-stress $\bar{\bar{\boldsymbol{\mu}}}$ second-order tensors. The relation of $\boldsymbol{\sigma}$ and $\bar{\bar{\boldsymbol{\mu}}}$ to $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ is discussed in Sect. 5. The true force and moment per unit area are defined by $\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}$ and $\mathbf{m} = \mathbf{n}\cdot\bar{\bar{\boldsymbol{\mu}}}$ and not by $\mathbf{P}$ and $\mathbf{R}$, which appear in (24) and (26) (see Mindlin and Eshel [36], p. 124).

We consider next the traction boundary value problem in which $\partial\mathcal{B}_u = \partial\mathcal{B}_v = \varnothing$ and $\partial\mathcal{B}_P = \partial\mathcal{B}_R = \partial\mathcal{B}$. Let $(\mathbf{u}^{(0)}, \boldsymbol{\varepsilon}^{(0)}, \boldsymbol{\tau}^{(0)})$ be the solution of the classical elasticity problem ($\ell = 0$) that corresponds to loads $\mathbf{b}$ and $\mathbf{P}$, i.e., the solution of the following boundary value problem

$$\nabla\cdot\boldsymbol{\tau}^{(0)} + \mathbf{b} = \mathbf{0}, \tag{28}$$

$$\boldsymbol{\varepsilon}^{(0)} = \frac{1}{2}\left(\mathbf{u}^{(0)}\nabla + \nabla\mathbf{u}^{(0)}\right), \tag{29}$$

$$\boldsymbol{\tau}^{(0)} = 2G\left(\boldsymbol{\varepsilon}^{(0)} + \frac{\nu}{1-2\nu}\varepsilon_{kk}^{(0)}\boldsymbol{\delta}\right), \quad \text{and} \tag{30}$$

$$\mathbf{n}\cdot\boldsymbol{\tau}^{(0)} = \mathbf{P} \quad \text{on } \partial\mathcal{B}. \tag{31}$$

We introduce the "gradient correction" $(\mathbf{u}^{(1)}, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\mu}^{(1)})$ so that the solution of the boundary value problem defined by (20)–(26) is written as

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} + \boldsymbol{\varepsilon}^{(1)}, \qquad \boldsymbol{\tau} = \boldsymbol{\tau}^{(0)} + \boldsymbol{\tau}^{(1)}, \qquad \boldsymbol{\mu} = \boldsymbol{\mu}^{(0)} + \boldsymbol{\mu}^{(1)}, \tag{32}$$

where $\boldsymbol{\mu}^{(0)} = \ell^2 \nabla \boldsymbol{\tau}^{(0)}$. Then, the gradient correction is defined by the following boundary value problem:

$$\nabla \cdot \left(\boldsymbol{\tau}^{(1)} - \nabla \cdot \boldsymbol{\mu}^{(1)}\right) = \nabla \cdot \boldsymbol{\mu}^{(0)}, \tag{33}$$

$$\boldsymbol{\varepsilon}^{(1)} = \frac{1}{2}\left(\mathbf{u}^{(1)}\nabla + \nabla \mathbf{u}^{(1)}\right), \tag{34}$$

$$\boldsymbol{\tau}^{(1)} = 2G\left(\boldsymbol{\varepsilon}^{(1)} + \frac{\nu}{1-2\nu}\varepsilon_{kk}^{(1)}\boldsymbol{\delta}\right), \quad \boldsymbol{\mu}^{(1)} = \ell^2 \nabla^2 \boldsymbol{\tau}^{(1)}, \tag{35}$$

and

$$\begin{aligned} &\mathbf{n} \cdot \left(\boldsymbol{\tau}^{(1)} - \nabla \cdot \boldsymbol{\mu}^{(1)}\right) - \mathbf{D} \cdot \left(\mathbf{n} \cdot \boldsymbol{\mu}^{(1)}\right) + (\mathbf{D} \cdot \mathbf{n})\,\mathbf{n}\left(\mathbf{n} \cdot \boldsymbol{\mu}^{(1)}\right) \\ &\quad = \mathbf{n} \cdot \left(\nabla \cdot \boldsymbol{\mu}^{(0)}\right) + \mathbf{D} \cdot \left(\mathbf{n} \cdot \boldsymbol{\mu}^{(0)}\right) - (\mathbf{D} \cdot \mathbf{n})\,\mathbf{n}\left(\mathbf{n} \cdot \boldsymbol{\mu}^{(0)}\right) \quad \text{on } \partial\mathcal{B}, \end{aligned} \tag{36}$$

$$\mathbf{n} \cdot \left(\mathbf{n} \cdot \boldsymbol{\mu}^{(1)}\right) = \mathbf{R} - \mathbf{n} \cdot \left(\mathbf{n} \cdot \boldsymbol{\mu}^{(0)}\right) \quad \text{on } \partial\mathcal{B}. \tag{37}$$

Equations (33)–(37) show that if $\mathbf{R} = \mathbf{0}$, then the "loads" on the right hand side of (33), (36), and (37) are all proportional to $\boldsymbol{\mu}^{(0)}$ (which equals $\ell^2 \nabla \boldsymbol{\tau}^{(0)}$). Therefore, when $\mathbf{R} = \mathbf{0}$, if $\boldsymbol{\mu}^{(0)}$ is proportional to a parameter, then the gradient correction $(\mathbf{u}^{(1)}, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{\mu}^{(1)})$ is proportional to the same parameter. An application of this property will be discussed in Sect. 7.1, where the problem of an annulus loaded by internal and external pressure is solved.

5 The "True" Stress and the "True" Couple Stress

The quantities $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ have dimensions of stress and double-stress respectively and are defined by (4) and (5) as work-conjugate to $\boldsymbol{\varepsilon}$ and $\boldsymbol{\kappa}$ (Mindlin [34]). However, the physical interpretation of $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ is not clear. Mindlin and Eshel [36] established the relationship between $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ and the true stress tensor $\boldsymbol{\sigma}$ and the true couple stress tensor $\bar{\bar{\boldsymbol{\mu}}}$ as described briefly in the following.

Mindlin and Eshel [36] consider the standard true stress vector $\mathbf{t}$ and true couple-stress vector $\mathbf{m}$, defined the usual way as force and moment per unit area. Then, the standard argument of force and moment equilibrium of an infinitesimal tetrahedron leads to the well know relations

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\sigma} \quad \text{and} \quad \mathbf{m} = \mathbf{n} \cdot \bar{\bar{\boldsymbol{\mu}}}, \tag{38}$$

where $\mathbf{n}$ is the unit vector normal to the infinitesimal area, and $(\boldsymbol{\sigma}, \bar{\bar{\boldsymbol{\mu}}})$ are the usual stress and couple-stress second-order tensors.

The principles of conservation of linear and angular momentum lead to the well known equations

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \bar{\bar{\boldsymbol{\mu}}} + \mathbf{s} + \mathbf{c} = \mathbf{0}, \tag{39}$$

where $\mathbf{c}$ is the body moment per unit volume and $s_i = e_{ijk}\sigma_{jk}$.

Let $\boldsymbol{\Phi}$ be the double-force per unit volume. Mindlin and Eshel [36] identify the symmetric part of $\boldsymbol{\Phi}$ with the body double-force without moment per unit volume, and the anti-symmetric part of $\boldsymbol{\Phi}$ with the body moment per unit volume with its components defined as

$\Phi_{[ij]} = \frac{1}{2} e_{ijk} c_k$ or $c_i = e_{ijk} \Phi_{[jk]}$. Mindlin and Eshel [36] have shown that (see also Germain [23–25], Aravas and Giannakopoulos [6])

$$\boldsymbol{\sigma} = \boldsymbol{\tau} - \frac{2}{3}\boldsymbol{\mu} \cdot \nabla - \frac{1}{3}\nabla \cdot \boldsymbol{\mu} - \boldsymbol{\Phi} \quad \text{or} \quad \sigma_{ij} = \tau_{ij} - \frac{2}{3}\mu_{ijk,k} - \frac{1}{3}\mu_{kij,k} - \Phi_{ij}, \tag{40}$$

and

$$\bar{\bar{\mu}}_{ij} = \frac{2}{3}\mu_{pqi} e_{pqj}. \tag{41}$$

6 Plane Strain Problems

We consider the plane displacement field

$$\mathbf{u} = u_\alpha \left(x_1, x_2\right) \mathbf{e}_\alpha. \tag{42}$$

The corresponding strain $\boldsymbol{\varepsilon}$ and strain-gradient $\boldsymbol{\kappa}$ fields are of the form

$$\boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta \quad \text{and} \quad \boldsymbol{\kappa} = \nabla\boldsymbol{\varepsilon} = \kappa_{\alpha\beta\gamma}\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma, \tag{43}$$

with

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \quad \text{and} \quad \kappa_{\alpha\beta\gamma} = \varepsilon_{\beta\gamma,\alpha} = \frac{1}{2}\left(u_{\beta,\gamma\alpha} + u_{\gamma,\beta\alpha}\right). \tag{44}$$

The plane strain conditions

$$\varepsilon_{33} = \frac{1}{2\mu}\left(\tau_{33} - \frac{\nu}{1+\nu}\tau_{kk}\delta_{33}\right) = 0 \quad \text{and} \quad \kappa_{\alpha 33} = \frac{1}{2\ell^2\mu}\left(\mu_{\alpha 33} - \frac{\nu}{1+\nu}\mu_{\alpha pp}\delta_{33}\right) = 0, \tag{45}$$

imply that

$$\tau_{33} = \nu\tau_{\alpha\alpha} \quad \text{and} \quad \mu_{\alpha 33} = \nu\mu_{\alpha\beta\beta}, \tag{46}$$

and $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ are of the form

$$\boldsymbol{\tau} = \tau_{\alpha\beta}\mathbf{e}_\alpha\mathbf{e}_\beta + \nu\tau_{\alpha\alpha}\mathbf{e}_3\mathbf{e}_3 \quad \text{and} \quad \boldsymbol{\mu} = \mu_{\alpha\beta\gamma}\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{e}_\gamma + \nu\mu_{\alpha\beta\beta}\mathbf{e}_\alpha\mathbf{e}_3\mathbf{e}_3. \tag{47}$$

The constitutive equations can be written now in the form

$$\tau_{\alpha\beta} = 2G\left(\varepsilon_{\alpha\beta} + \frac{\nu}{1-2\nu}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right), \qquad \mu_{\alpha\beta\gamma} = 2\ell^2 G\left(\kappa_{\alpha\beta\gamma} + \frac{\nu}{1-2\nu}\kappa_{\alpha\delta\delta}\delta_{\beta\gamma}\right), \tag{48}$$

with

$$\tau_{\alpha\alpha} = \frac{2G}{1-2\nu}\varepsilon_{\alpha\alpha} \quad \text{and} \quad \mu_{\alpha\beta\beta} = \frac{2\ell^2 G}{1-2\nu}\kappa_{\alpha\beta\beta}. \tag{49}$$

The above equations can be inverted to yield

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\tau_{\alpha\beta} - \nu\tau_{\gamma\gamma}\delta_{\alpha\beta}\right), \qquad \kappa_{\alpha\beta\gamma} = \frac{1}{2\ell^2 G}\left(\mu_{\alpha\beta\gamma} - \nu\mu_{\alpha\delta\delta}\delta_{\beta\gamma}\right). \tag{50}$$

6.1 The Compatibility Equations

The compatibility equations (12) and (13) now take the form

$$\mathbf{S} = \left(2\varepsilon_{21,12} - \varepsilon_{11,22} - \varepsilon_{22,11}\right)\mathbf{e}_3\mathbf{e}_3 \\ = \frac{1}{2G}\left(2\tau_{21,12} - \tau_{11,22} - \tau_{22,11} - \nu\nabla^2\tau_{\alpha\alpha}\right)\mathbf{e}_3\mathbf{e}_3 = \mathbf{0}, \tag{51}$$

and

$$\mathbf{P} = \left(\kappa_{2\beta\gamma,1} - \kappa_{1\beta\gamma,2}\right)\mathbf{e}_3\mathbf{e}_\beta\mathbf{e}_\gamma \\ = \frac{1}{2\ell^2 G}\left[\mu_{2\beta\gamma,1} - \mu_{1\beta\gamma,2} - \nu\left(\mu_{2\alpha\alpha,1} - \mu_{1\alpha\alpha,2}\right)\delta_{\beta\gamma}\right]\mathbf{e}_3\mathbf{e}_\beta\mathbf{e}_\gamma = \mathbf{0}. \tag{52}$$

The above compatibility equations can be written in the form

$$2\tau_{21,12} - \tau_{11,22} - \tau_{22,11} - \nu\nabla^2\tau_{\alpha\alpha} = 0, \tag{53}$$

and

$$(1-\nu)(\mu_{211,1} - \mu_{111,2}) - \nu(\mu_{222,1} - \mu_{122,2}) = 0, \tag{54}$$

$$(1-\nu)(\mu_{222,1} - \mu_{122,2}) - \nu(\mu_{211,1} - \mu_{111,2}) = 0, \tag{55}$$

$$\mu_{212,1} - \mu_{112,2} = 0. \tag{56}$$

For $\nu \neq 1/2$, the κ-compatibility equations (54)–(56) are equivalent to

$$\mu_{211,1} - \mu_{111,2} = 0, \tag{57}$$

$$\mu_{222,1} - \mu_{122,2} = 0, \tag{58}$$

$$\mu_{212,1} - \mu_{112,2} = 0. \tag{59}$$

If the region of interest is $(N+1)$-tuply connected, the following additional $6N$ conditions should be satisfied on the closed curves C_n $(n = 1, 2, \ldots, N)$ introduced in Sect. 3 above:

$$S^u_\alpha = \oint_{C_n}\left(\varepsilon_{\alpha\beta} - x_\zeta e_{\alpha\zeta} e_{\gamma\delta}\varepsilon_{\gamma\beta,\delta}\right)dx_\beta \\ = \frac{1}{2G}\oint_{C_n}\left[\tau_{\alpha\beta} - \nu\tau_{\gamma\gamma}\delta_{\alpha\beta} - x_\zeta e_{\alpha\zeta} e_{\gamma\delta}\left(\tau_{\gamma\beta,\delta} - \nu\tau_{\eta\eta,\delta}\delta_{\gamma\beta}\right)\right] = 0, \quad n = 1, 2, \ldots, N, \tag{60}$$

$$S^\omega_3 = \oint_{C_n} e_{\beta\gamma}\varepsilon_{\beta\alpha,\gamma}\,dx_\alpha = \frac{1}{2G}\oint_{C_n} e_{\beta\gamma}\left(\tau_{\alpha\beta,\gamma} - \nu\tau_{\delta\delta,\gamma}\delta_{\alpha\beta}\right)dx_\alpha, \quad n = 1, 2, \ldots, N, \tag{61}$$

$$P^e_{\alpha\beta} = \oint_{C_n}\kappa_{\gamma\alpha\beta}\,dx_\gamma = \frac{1}{2G}\oint_{C_n}\left(\tau_{\alpha\beta,\gamma} - \nu\tau_{\delta\delta,\gamma}\delta_{\alpha\beta}\right)dx_\gamma, \quad n = 1, 2, \ldots, N. \tag{62}$$

6.2 The Plane-Strain Traction Boundary Value Problem in Terms of $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$

For simplicity we assume for the rest of the paper that the body forces vanish ($\mathbf{b} = \mathbf{0}$). Then the plane strain boundary value problem becomes

$$\left(\tau_{\alpha\beta} - \mu_{\gamma\alpha\beta,\gamma}\right)_{,\beta} = 0, \tag{63}$$

$$2\tau_{21,12} - \tau_{11,22} - \tau_{22,11} - \nu\nabla^2\tau_{\alpha\alpha} = 0, \tag{64}$$

$$\mu_{\gamma\alpha\beta} = \ell^2\tau_{\alpha\beta,\gamma}, \tag{65}$$

and

$$n_\beta\left(\tau_{\beta\alpha} - \mu_{\gamma\beta\alpha,\gamma}\right) - D_\beta\left(n_\gamma\mu_{\gamma\beta\alpha}\right) + \left(D_\delta n_\delta\right) n_\gamma n_\beta \mu_{\gamma\beta\alpha} = P_\alpha \quad \text{on } \partial\mathcal{B}, \tag{66}$$

$$n_\gamma n_\beta \mu_{\gamma\beta\alpha} = R_\alpha \quad \text{on } \partial\mathcal{B}, \tag{67}$$

where $\mathcal{B}$ is now a two-dimensional domain and $\partial\mathcal{B}$ its smooth boundary.

Equations (63)–(65) form a linear second-order system of equations, which together with the boundary conditions (66) and (67), define $(\tau_{11}, \tau_{22}, \tau_{12})$ and $(\mu_{111}, \mu_{122}, \mu_{112}, \mu_{211}, \mu_{222}, \mu_{212})$.

We can eliminate the double-stress $\boldsymbol{\mu}$ from the above equations and restate the boundary value problem in the following form:

$$\left(\tau_{\alpha\beta} - \ell^2\nabla^2\tau_{\alpha\beta}\right)_{,\beta} = 0, \tag{68}$$

$$2\tau_{21,12} - \tau_{11,22} - \tau_{22,11} - \nu\nabla^2\tau_{\alpha\alpha} = 0 \tag{69}$$

and

$$n_\beta\left(\tau_{\alpha\beta} - \ell^2\nabla^2\tau_{\alpha\beta}\right) - \ell^2 D_\beta\left(n_\gamma\tau_{\alpha\beta,\gamma}\right) + \ell^2\left(D_\delta n_\delta\right) n_\gamma n_\beta \tau_{\alpha\beta,\gamma} = P_\alpha \quad \text{on } \partial\mathcal{B}, \tag{70}$$

$$\ell^2 n_\gamma n_\beta \tau_{\alpha\beta,\gamma} = R_\alpha \quad \text{on } \partial\mathcal{B}. \tag{71}$$

Equations (68) and (69) form a linear third-order system of equations, which together with the boundary conditions (70) and (71), define the three unknowns $(\tau_{11}, \tau_{22}, \tau_{12})$.

If the region of interest is $(N+1)$-tuply connected, the following additional conditions should be satisfied on the closed curves C_n $(n = 1, 2, \ldots, N)$ introduced in Sect. 3 above:

$$S^u_\alpha = \frac{1}{2G}\oint_{C_n}\left[\tau_{\alpha\beta} - \nu\tau_{\gamma\gamma}\delta_{\alpha\beta} - x_\zeta e_{\alpha\zeta} e_{\gamma\delta}\left(\tau_{\gamma\beta,\delta} - \nu\tau_{\eta\eta,\delta}\delta_{\gamma\beta}\right)\right] = 0, \tag{72}$$

$$S^\omega_3 = \frac{1}{2G}\oint_{C_n} e_{\beta\gamma}\left(\tau_{\alpha\beta,\gamma} - \nu\tau_{\delta\delta,\gamma}\delta_{\alpha\beta}\right) dx_\alpha, \tag{73}$$

$$P^e_{\alpha\beta} = \frac{1}{2G}\oint_{C_n}\left(\tau_{\alpha\beta,\gamma} - \nu\tau_{\delta\delta,\gamma}\delta_{\alpha\beta}\right) dx_\gamma. \tag{74}$$

6.3 Stress and Double-Stress Functions

Schaefer [40], Mindlin [33] and Carlson [10] derived the appropriate stress functions for plane problems with couple stresses. For the particular case of a homogeneous, isotropic, linearly elastic constrained-Cosserat material, they used the equilibrium and the κ-compatibility conditions to show that the stress and couple-stress components can be written in terms of two stress functions (see also Boresi and Chong [8], pp. 390–398).

We follow a similar approach here, starting with the κ-compatibility conditions (57)–(59):

$$\mu_{211,1} = \mu_{111,2}, \qquad \mu_{222,1} = \mu_{122,2}, \qquad \mu_{212,1} = \mu_{112,2}. \tag{75}$$

According to the theory of total differentials (Courant and John [14]), the above equations imply the existence of functions F_{11}, F_{22}, and F_{12} (single valued if $\mathcal{B}$ is simply connected) such that

$$\mu_{111} = \frac{\partial F_{11}}{\partial x_1}, \qquad \mu_{211} = \frac{\partial F_{11}}{\partial x_2}, \tag{76}$$

$$\mu_{122} = \frac{\partial F_{22}}{\partial x_1}, \qquad \mu_{222} = \frac{\partial F_{22}}{\partial x_2}, \tag{77}$$

$$\mu_{112} = \frac{\partial F_{12}}{\partial x_1}, \qquad \mu_{212} = \frac{\partial F_{12}}{\partial x_2}. \tag{78}$$

The above equations can be written compactly in the form

$$\mu_{\alpha\beta\gamma} = \frac{\partial F_{\beta\gamma}}{\partial x_\alpha} \quad \text{or } \boldsymbol{\mu} = \nabla \mathbf{F}, \tag{79}$$

with $F_{21} = F_{12}$. Comparison of (79) with the constitutive equation (7), i.e.,

$$\boldsymbol{\mu} = \ell^2 \nabla \boldsymbol{\tau} \tag{80}$$

leads to the conclusion that $\mathbf{F}$ can be identified with $\ell^2 \boldsymbol{\tau}$, i.e., the second order tensor $\boldsymbol{\tau}$ can be viewed as a double-stress function in the present model.

We consider next equations (68):

$$\left(\tau_{\alpha\beta} - \ell^2 \nabla^2 \tau_{\alpha\beta}\right)_{,\beta} = 0, \tag{81}$$

which imply the existence of the well-known Airy [5] stress function f (single valued if $\mathcal{B}$ is simply connected) such that

$$\tau_{11} - \ell^2 \nabla^2 \tau_{11} = f_{,22}, \qquad \tau_{22} - \ell^2 \nabla^2 \tau_{22} = f_{,11}, \qquad \tau_{12} - \ell^2 \nabla^2 \tau_{12} = -f_{,12}. \tag{82}$$

The above equations can be written in compact form as

$$\tau_{\alpha\beta} - \ell^2 \nabla^2 \tau_{\alpha\beta} = \nabla^2 f \delta_{\alpha\beta} - f_{,\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} f_{,\gamma\delta}, \tag{83}$$

where $e_{\alpha\beta}$ is the alternating symbol in two dimensions that takes the values $e_{11} = e_{22} = 0$ and $e_{12} = -e_{21} = 1$.

It should be noted that (82) are a consequence of (81) and are independent of any constitutive equations $\boldsymbol{\tau}$ and $\boldsymbol{\mu}$ may be required to satisfy. However, (79) are valid only for the particular gradient elasticity model described in Sect. 2. Günther [26], Schaefer [40], and Carlson [11–13] presented stress functions for couple and dipolar stresses appropriate for all materials.

The plane-strain boundary value problem can be written now in terms of τ_{ij} and the stress function f as follows

$$\tau_{11} - \ell^2 \nabla^2 \tau_{11} = f_{,22}, \tag{84}$$

$$\tau_{22} - \ell^2 \nabla^2 \tau_{22} = f_{,11}, \tag{85}$$

$$\tau_{12} - \ell^2 \nabla^2 \tau_{12} = -f_{,12}, \tag{86}$$

$$2\tau_{21,12} - \tau_{11,22} - \tau_{22,11} - \nu \nabla^2 \tau_{\alpha\alpha} = 0, \tag{87}$$

and

$$n_\beta \left(\nabla^2 f \delta_{\alpha\beta} - f_{,\alpha\beta} \right) - \ell^2 D_\beta \left(n_\gamma \tau_{\alpha\beta,\gamma} \right) + \ell^2 \left(D_\delta n_\delta \right) n_\gamma n_\beta \tau_{\alpha\beta,\gamma} = P_\alpha \quad \text{on } \partial\mathcal{B}, \tag{88}$$

$$\ell^2 n_\gamma n_\beta \tau_{\alpha\beta,\gamma} = R_\alpha \quad \text{on } \partial\mathcal{B}. \tag{89}$$

Equations (84)–(87) form a linear second-order system of equations, which together with the boundary conditions (88) and (89), define the four unknowns $(f, \tau_{11}, \tau_{22}, \tau_{12})$. The stress function f is defined to within a linear function of the form $k_\alpha x_\alpha + c$ that causes no stress.

If the domain is multiply connected, conditions (72)–(74) are imposed as well.

If polar coordinates are used, equations (84)–(87) have the form

$$\tau_{rr} - \ell^2 \left\{ \frac{\partial^2 \tau_{rr}}{\partial r^2} + \frac{1}{r} \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2 \tau_{rr}}{\partial \theta^2} - 4 \frac{\partial \tau_{r\theta}}{\partial \theta} - 2 \left(\tau_{rr} - \tau_{\theta\theta} \right) \right] \right\} = \frac{1}{r} \frac{\partial f}{\partial r} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}, \tag{90}$$

$$\tau_{\theta\theta} - \ell^2 \left\{ \frac{\partial^2 \tau_{\theta\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2 \tau_{\theta\theta}}{\partial \theta^2} + 4 \frac{\partial \tau_{r\theta}}{\partial \theta} + 2 \left(\tau_{rr} - \tau_{\theta\theta} \right) \right] \right\} = \frac{\partial^2 f}{\partial r^2}, \tag{91}$$

$$\tau_{r\theta} - \ell^2 \left\{ \frac{\partial^2 \tau_{r\theta}}{\partial r^2} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r^2} \left[\frac{\partial^2 \tau_{r\theta}}{\partial \theta^2} + 2 \frac{\partial}{\partial \theta} \left(\tau_{rr} - \tau_{\theta\theta} \right) - 4 \tau_{r\theta} \right] \right\} = -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial f}{\partial \theta} \right), \tag{92}$$

$$-\frac{\partial^2 \tau_{\theta\theta}}{\partial r^2} + \frac{2}{r} \frac{\partial^2 \tau_{r\theta}}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial}{\partial r} \left(\tau_{rr} - 2\tau_{\theta\theta} \right) + \frac{2}{r^2} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{1}{r^2} \frac{\partial^2 \tau_{rr}}{\partial \theta^2} + \nu \nabla^2 \left(\tau_{rr} + \tau_{\theta\theta} \right) = 0. \tag{93}$$

The boundary conditions (88) and (89) can be stated in polar coordinated, if we take into account that

$$\nabla = \mathbf{e}_r \frac{\partial}{\partial r} + \mathbf{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta}, \qquad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \tag{94}$$

$$\mathbf{D} = \mathbf{e}_r n_\theta \left(n_\theta \frac{\partial}{\partial r} - n_r \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \mathbf{e}_\theta n_r \left(\frac{n_r}{r} \frac{\partial}{\partial \theta} - n_\theta \frac{\partial}{\partial r} \right), \tag{95}$$

and

$$\begin{aligned} \nabla \boldsymbol{\tau} = {} & \frac{\partial \tau_{rr}}{\partial r} \mathbf{e}_r \mathbf{e}_r \mathbf{e}_r + \frac{\partial \tau_{\theta\theta}}{\partial r} \mathbf{e}_r \mathbf{e}_\theta \mathbf{e}_\theta + \frac{\partial \tau_{r\theta}}{\partial r} \mathbf{e}_r \left(\mathbf{e}_r \mathbf{e}_\theta + \mathbf{e}_\theta \mathbf{e}_r \right) \\ & + \frac{1}{r} \left(\frac{\partial \tau_{rr}}{\partial \theta} - 2\tau_{r\theta} \right) \mathbf{e}_\theta \mathbf{e}_r \mathbf{e}_r + \frac{1}{r} \left(\frac{\partial \tau_{\theta\theta}}{\partial \theta} + 2\tau_{r\theta} \right) \mathbf{e}_\theta \mathbf{e}_\theta \mathbf{e}_\theta \\ & + \frac{1}{r} \left(\frac{\partial \tau_{r\theta}}{\partial \theta} + \tau_{rr} - \tau_{\theta\theta} \right) \mathbf{e}_\theta \left(\mathbf{e}_r \mathbf{e}_\theta + \mathbf{e}_\theta \mathbf{e}_r \right). \end{aligned} \tag{96}$$

In particular, on a circular arc of radius r, $\mathbf{n} = \pm\mathbf{e}_r$ $(n_r = \pm 1, n_\theta = 0)$ and

$$\mathbf{D} = \mathbf{e}_\theta \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \mathbf{D}\cdot\mathbf{n} = D_\alpha n_\alpha = \pm\frac{1}{r}. \tag{97}$$

The polar components of (72)–(74), required in multiply connected domains, are given in Appendix A.

6.4 Axisymmetric Problems

We consider plane-strain axisymmetric problems, in which the solution is independent of θ and

$$u_\theta = 0, \qquad \tau_{r\theta} = 0. \tag{98}$$

Equations (90)–(93) reduce now to

$$-\ell^2\frac{d^2\tau_{rr}}{dr^2} - \frac{\ell^2}{r}\frac{d\tau_{rr}}{dr} + \left(1 + 2\frac{\ell^2}{r^2}\right)\tau_{rr} - 2\frac{\ell^2}{r^2}\tau_{\theta\theta} = \frac{1}{r}\frac{df}{dr}, \tag{99}$$

$$-\ell^2\frac{d^2\tau_{\theta\theta}}{dr^2} - \frac{\ell^2}{r}\frac{d\tau_{\theta\theta}}{dr} + \left(1 + 2\frac{\ell^2}{r^2}\right)\tau_{\theta\theta} - 2\frac{\ell^2}{r^2}\tau_{rr} = \frac{d^2 f}{dr^2}, \tag{100}$$

$$\nu\frac{d^2\tau_{rr}}{dr^2} - (1-\nu)\frac{d^2\tau_{\theta\theta}}{dr^2} + \frac{1+\nu}{r}\frac{d\tau_{rr}}{dr} - \frac{2-\nu}{r}\frac{d\tau_{\theta\theta}}{dr} = 0. \tag{101}$$

On a circular boundary of radius r with outward unit normal $\mathbf{n} = \pm\mathbf{e}_r$, the boundary conditions (88) and (89) become

$$\pm\frac{1}{r}\left(\frac{df}{dr} + \ell^2\frac{d\tau_{\theta\theta}}{dr}\right) = P_r \quad \text{and} \quad \ell^2\frac{d\tau_{rr}}{dr} = R_r \quad \text{on } \partial\mathcal{B}. \tag{102}$$

Equations (99)–(102) form a second-order system of linear ordinary differential equations which defines the Airy stress function f and the "double stress functions" τ_{rr} and $\tau_{\theta\theta}$.

If the domain is multiply connected, conditions (72)–(74) should be imposed as well; the form of these conditions on circular boundaries for axisymmetric problems is given in Appendix A.

In the following we present the general solution of the aforementioned system of equations. The form of (99) and (100) suggests that the problem can be simplified if we add and subtract them. In fact, if we define

$$S = \tau_{rr} + \tau_{\theta\theta}, \qquad D = \tau_{rr} - \tau_{\theta\theta}, \quad \text{and} \quad F = \frac{1}{r}\frac{df}{dr}, \tag{103}$$

so that

$$\tau_{rr} = \frac{S+D}{2}, \qquad \tau_{\theta\theta} = \frac{S-D}{2}, \quad \text{and} \quad f = \int rF\,dr, \tag{104}$$

addition and subtraction of (99) and (100) yield

$$-\ell^2\frac{d^2S}{dr^2} - \frac{\ell^2}{r}\frac{dS}{dr} + S = \frac{1}{r}\frac{d}{dr}\left(r^2F\right) \tag{105}$$

and

$$\ell^2 \frac{d^2 D}{dr^2} - \frac{\ell^2}{r}\frac{dD}{dr} + \left(1 + 4\frac{\ell^2}{r^2}\right) D = -r\frac{dF}{dr}, \tag{106}$$

whereas (101) becomes

$$(1-2\nu)\frac{d}{dr}\left(r\frac{dS}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^3\frac{dD}{dr}\right). \tag{107}$$

Elimination of D and F from (105)–(107) leads to the following fourth-order ordinary differential equation for S

$$\ell^2\frac{d^4 S}{dr^4} + \frac{2\ell^2}{r}\frac{d^3 S}{dr^3} - \left(1 + \frac{\ell^2}{r^2}\right)\frac{d^2 S}{dr^2} - \frac{1}{r}\left(1 - \frac{\ell^2}{r^2}\right)\frac{dS}{dr} = 0, \tag{108}$$

and rest of the solution is defined by

$$\frac{dD}{dr} = \frac{1}{r^3}\left[(1-2\nu)\int r^2\frac{d}{dr}\left(r\frac{dS}{dr}\right)dr + c_5\right], \tag{109}$$

$$D = \frac{1}{2}\left[2\ell^2(1-\nu)\left(r\frac{d^3 S}{dr^3} + \frac{d^2 S}{dr^2} - \frac{1}{r}\frac{dS}{dr}\right) - r\left(\frac{dS}{dr} + \frac{dD}{dr}\right)\right], \tag{110}$$

$$\tau_{rr} = \frac{S+D}{2}, \qquad \tau_{\theta\theta} = \frac{S-D}{2}, \tag{111}$$

$$F = -\ell^2\frac{d\tau_{rr}^2}{dr^2} - 2\frac{\ell^2}{r}\frac{d\tau_{rr}}{dr} + \tau_{rr} + 2\frac{\ell^2}{r^2}(\tau_{rr} - \tau_{\theta\theta}), \tag{112}$$

$$f = \int r F\, dr. \tag{113}$$

where c_5 is an arbitrary constant.

The general solution of (108) is

$$S(r) = c_1 + c_2 K_0\left(\frac{r}{\ell}\right) + c_3 I_0\left(\frac{r}{\ell}\right) + c_4 \ln\frac{r}{\ell}, \tag{114}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants, and (I_n, K_n) are modified Bessel functions of the first and second kind. The rest of the solution is determined by (109)–(113):

$$\tau_{rr}(r) = \frac{1}{2}\left(c_1 - \frac{c_6}{r^2}\right) + \frac{c_2}{2}\left[K_0\left(\frac{r}{\ell}\right) + (1-2\nu)K_1\left(\frac{r}{\ell}\right)\right] + \frac{c_3}{2}\left[I_0\left(\frac{r}{\ell}\right) + (1-2\nu)I_1\left(\frac{r}{\ell}\right)\right] + \frac{c_4}{2}\left(\ln\frac{r}{\ell} - \frac{1}{2}\right), \tag{115}$$

$$\tau_{\theta\theta}(r) = \frac{1}{2}\left(c_1 + \frac{c_6}{r^2}\right) + \frac{c_2}{2}\left[K_0\left(\frac{r}{\ell}\right) - (1-2\nu)K_1\left(\frac{r}{\ell}\right)\right] + \frac{c_3}{2}\left[I_0\left(\frac{r}{\ell}\right) - (1-2\nu)I_1\left(\frac{r}{\ell}\right)\right] + \frac{c_4}{2}\left(\ln\frac{r}{\ell} + \frac{1}{2}\right), \tag{116}$$

$$f(r) = \frac{c_1}{4}r^2 - \frac{c_6}{2}\ln r - c_4\ell^2\left[\ln r + \frac{r^2}{4\ell^2}\left(1 - \ln\frac{r}{\ell}\right)\right], \tag{117}$$

where $c_6 = c_5 + 4(1-2\nu)c_2\ell^2$.

The terms that involve c_4 in the above solution are not consistent with the assumption of an axisymmetric solution;[2] therefore we set $c_4 = 0$.

The corresponding displacement is

$$u_r(r) = \frac{1}{2G}\left\{\frac{1-2\nu}{2}c_1 r + \frac{c_6}{2r} - (1-2\nu)\,\ell\left[c_2 K_1\left(\frac{r}{\ell}\right) - c_3 I_1\left(\frac{r}{\ell}\right)\right]\right\}. \tag{119}$$

Finally, the boundary conditions (102) on a circular boundary with outward unit normal $\mathbf{n} = \pm\mathbf{e}_r$ become

$$P_r(r) = \pm\left\{\frac{1}{2}\left(c_1 - \frac{c_6}{r^2}\right) - c_2\frac{\ell}{r}\left[\nu K_1\left(\frac{r}{\ell}\right) - (1-2\nu)\,K_2\left(\frac{r}{\ell}\right)\right]\right.$$
$$\left. + c_3\frac{\ell}{r}\left[\nu I_1\left(\frac{r}{\ell}\right) + (1-2\nu)\,I_2\left(\frac{r}{\ell}\right)\right] - \frac{c_6}{2}\frac{\ell^2}{r^4}\right\}, \tag{120}$$

and

$$R_r(r) = -c_2\ell\left[(1-\nu)\,K_1\left(\frac{r}{\ell}\right) + (1-2\nu)\,K_2\left(\frac{r}{\ell}\right)\right]$$
$$+ c_3\ell\left[(1-\nu)\,I_1\left(\frac{r}{\ell}\right) - (1-2\nu)\,I_2\left(\frac{r}{\ell}\right)\right] + c_6\frac{\ell^2}{r^3}. \tag{121}$$

Assuming that the double-force per unit volume vanishes, i.e. $\boldsymbol{\Phi} = \mathbf{0}$, and using (40) and (41), we conclude that the only non-zero in-plane components the double-stress $\boldsymbol{\mu}$, the true stress $\boldsymbol{\sigma}$, and true couple stress $\bar{\bar{\boldsymbol{\mu}}}$ are as follows (see also Appendix A in Aravas and Giannakopoulos [6])

$$\mu_{rrr} = \ell^2\frac{d\tau_{rr}}{dr}, \qquad \mu_{r\theta\theta} = \ell^2\frac{d\tau_{\theta\theta}}{dr}, \qquad \mu_{\theta r\theta} = \mu_{\theta\theta r} = \ell^2\frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \tag{122}$$

$$\sigma_{rr} = \tau_{rr} - \frac{d\mu_{rrr}}{dr} - \frac{1}{r}\left(\mu_{rrr} - \frac{2}{3}\mu_{r\theta\theta} - \frac{4}{3}\mu_{\theta r\theta}\right), \tag{123}$$

$$\sigma_{\theta\theta} = \tau_{\theta\theta} - \frac{1}{3}\frac{d\mu_{r\theta\theta}}{dr} - \frac{2}{3}\frac{d\mu_{\theta r\theta}}{dr} - \frac{\mu_{r\theta\theta} + 2\mu_{\theta r\theta}}{r}, \tag{124}$$

and

$$\bar{\bar{\mu}}_{\theta 3} = -\bar{\bar{\mu}}_{3\theta} = \frac{2}{3}\left(\mu_{r\theta\theta} - \mu_{\theta r\theta}\right). \tag{125}$$

[2]The strains corresponding to the c_4-terms are

$$\varepsilon_{rr} = -\frac{c_4}{4G}\left[1 - 2(1-2\nu)\ln r\right] \quad \text{and} \quad \varepsilon_{\theta\theta} = \frac{c_4}{4G}\left[1 + 2(1-2\nu)\ln r\right]. \tag{118}$$

In axisymmetric problems with $u_\theta = 0$, the radial and hoop strains are $\varepsilon_{rr} = du_r/dr$ and $\varepsilon_{\theta\theta} = u_r/r$. Therefore, if we evaluate $u_r = r\varepsilon_{\theta\theta}$ and substitute it into $\varepsilon_{rr} = du_r/dr$, we conclude that $c_4 = 0$. The c_4-terms are known to correspond to u_θ displacements that are not single-valued and cannot be used in multiply connected domains (such as an annulus) expect in those special problems involving "dislocations" (see following Sect. 7.1 and Soutas-Little [41], p. 159).

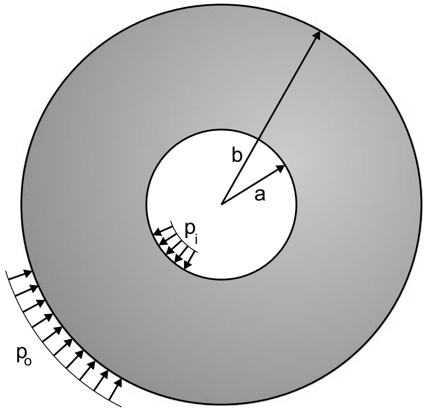


Fig. 2 Annulus loaded by internal and external pressure

7 Examples

7.1 An Annulus Subjected to Internal and External Pressure

We consider the problem of an annulus loaded by an internal pressure p_i and an external pressure p_o under conditions of plane strain (Fig. 2).

The domain is doubly connected; therefore, the conditions (72)–(74) should be imposed on the inner and outer circular boundaries of the annulus. These conditions together with the strain compatibility equation (101) guarantee that the corresponding displacement field is single valued in the doubly connected domain. As discussed in Appendix A, conditions (72) and (74) are satisfied identically, whereas (73) requires that the following condition be satisfied

$$\left[-\nu\frac{d\tau_{rr}}{dr}+(1-\nu)\frac{d\tau_{\theta\theta}}{dr}-\frac{\tau_{rr}-\tau_{\theta\theta}}{r}\right]_{r=a}=0. \tag{126}$$

Using (115) and (116) for τ_{rr} and $\tau_{\theta\theta}$ in the above equation, we find

$$c_4\frac{1-\nu}{a}=0, \tag{127}$$

i.e., it is required that $c_4=0$ for the displacement field to be single valued. This finding is consistent with the remark made immediately after (116) concerning the vanishing of c_4.

In the case of the classical "local" linear isotropic elasticity ($\ell=0$), the solution is of the form

$$f^{(0)}=\frac{1}{2}Ar^2+B\ln r, \qquad \sigma_{rr}^{(0)}=\tau_{rr}^{(0)}=A+\frac{B}{r^2}, \qquad \sigma_{\theta\theta}^{(0)}=\tau_{\theta\theta}^{(0)}=A-\frac{B}{r^2}, \tag{128}$$

and

$$u_r^{(0)}=\frac{1}{2G}\left[(1-2\nu)\,Ar-\frac{B}{r}\right], \tag{129}$$

where

$$A=\frac{p_ia^2-p_ob^2}{b^2-a^2}, \qquad B=(p_o-p_i)\,\frac{a^2b^2}{b^2-a^2}, \tag{130}$$

and (p_i, p_o) are the inner and outer applied pressure loads as shown in Fig. 2.

In order to make connection with the aforementioned "local" solution, we set

$$c_1 = c_7 + 2A, \qquad c_6 = c_8 - 2B, \tag{131}$$

so that the gradient elasticity solution is written in the form

$$f(r) = f^{(0)}(r) + \frac{c_7}{4} r^2 - \frac{c_8}{2} \ln r, \tag{132}$$

$$\begin{aligned} \tau_{rr}(r) = \tau_{rr}^{(0)}(r) + \frac{1}{2}\left(c_7 - \frac{c_8}{r^2}\right) + \frac{c_2}{2}\left[K_0\left(\frac{r}{\ell}\right) + (1-2\nu)\,K_1\left(\frac{r}{\ell}\right)\right] \\ + \frac{c_3}{2}\left[I_0\left(\frac{r}{\ell}\right) + (1-2\nu)\,I_1\left(\frac{r}{\ell}\right)\right], \end{aligned} \tag{133}$$

$$\begin{aligned} \tau_{\theta\theta}(r) = \tau_{\theta\theta}^{(0)}(r) + \frac{1}{2}\left(c_7 + \frac{c_8}{r^2}\right) + \frac{c_2}{2}\left[K_0\left(\frac{r}{\ell}\right) - (1-2\nu)\,K_1\left(\frac{r}{\ell}\right)\right] \\ + \frac{c_3}{2}\left[I_0\left(\frac{r}{\ell}\right) - (1-2\nu)\,I_1\left(\frac{r}{\ell}\right)\right], \end{aligned} \tag{134}$$

and

$$\begin{aligned} \varepsilon_{rr}(r) = \varepsilon_{rr}^{(0)}(r) + \frac{1}{2G}\Bigg\{\frac{1-2\nu}{2}c_7 - \frac{c_8}{2r^2} \\ + (1-2\nu)\left\{c_2\left[K_0\left(\frac{r}{\ell}\right) + \frac{\ell}{r}K_1\left(\frac{r}{\ell}\right)\right] + c_3\left[I_0\left(\frac{r}{\ell}\right) - \frac{\ell}{r}I_1\left(\frac{r}{\ell}\right)\right]\right\}\Bigg\}, \end{aligned} \tag{135}$$

$$\begin{aligned} \varepsilon_{\theta\theta}(r) = \varepsilon_{\theta\theta}^{(0)}(r) + \frac{1}{2G}\Bigg\{\frac{1-2\nu}{2}c_7 + \frac{c_8}{2r^2} \\ - (1-2\nu)\frac{\ell}{r}\left[c_2 K_1\left(\frac{r}{\ell}\right) - c_3 I_1\left(\frac{r}{\ell}\right)\right]\Bigg\}, \end{aligned} \tag{136}$$

$$\begin{aligned} u_r(r) = r\varepsilon_{\theta\theta} = u_r^{(0)}(r) + \frac{1}{2G}\Bigg\{\frac{1-2\nu}{2}c_7 r + \frac{c_8}{2r} \\ - (1-2\nu)\,\ell\left[c_2 K_1\left(\frac{r}{\ell}\right) - c_3 I_1\left(\frac{r}{\ell}\right)\right]\Bigg\}. \end{aligned} \tag{137}$$

The only non-zero in-plane components of the true stress $\boldsymbol{\sigma}$ and the true couple-stress $\bar{\bar{\boldsymbol{\mu}}}$ are

$$\sigma_{rr}(r) = \sigma_{rr}^{(0)}(r) + \frac{1}{2}\left(c_7 - \frac{c_8}{r^2}\right) - \frac{2\nu}{3}\frac{\ell}{r}\left[c_2 K_1\left(\frac{r}{\ell}\right) - c_3 I_1\left(\frac{r}{\ell}\right)\right], \tag{138}$$

$$\begin{aligned} \sigma_{\theta\theta}(r) = \sigma_{\theta\theta}^{(0)}(r) + \frac{1}{2}\left(c_7 + \frac{c_8}{r^2}\right) + \frac{2\nu}{3}\Bigg\{c_2\left[K_0\left(\frac{r}{\ell}\right) + \frac{\ell}{r}K_1\left(\frac{r}{\ell}\right)\right] \\ + c_3\left[I_0\left(\frac{r}{\ell}\right) - \frac{\ell}{r}I_1\left(\frac{r}{\ell}\right)\right]\Bigg\}, \end{aligned} \tag{139}$$

and

$$\bar{\bar{\mu}}_{3\theta}(r) = -\bar{\bar{\mu}}_{\theta 3}(r) = \frac{2\nu}{3}\ell\left[c_2 K_1\left(\frac{r}{\ell}\right) - c_3 I_1\left(\frac{r}{\ell}\right)\right]. \tag{140}$$

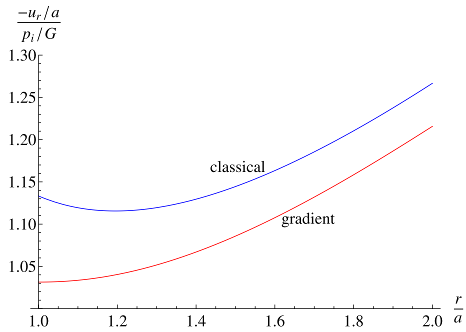


Fig. 3 Spatial variation of normalized displacement $(|u_r|/a)/(p_i/G)$ for $\nu = 0.3$, $\ell/a = 0.2$, $b = 2a$, and $p_o = 2p_i$

The terms that involve c_2, c_3, c_7, and c_8 define now the "gradient correction", where the constants c_2, c_3, c_7, and c_8 are determined from the following boundary conditions:

$$\text{on} \quad r = a \quad (\mathbf{n} = -\mathbf{e}_r): \quad P_r(a) = p_i, \qquad R_r(a) = 0, \tag{141}$$

$$\text{on} \quad r = b \quad (\mathbf{n} = \mathbf{e}_r): \quad P_r(b) = -p_o, \qquad R_r(b) = 0. \tag{142}$$

The boundary conditions (141) and (142) define a system of four algebraic linear equations that is solved for c_2, c_3, c_7 and c_8. The resulting expressions for c_2, c_3, c_7 and c_8 are lengthy and are listed in Appendix B.

We note that the solution is of the form

$$f(r) = f^{(0)}(r) + (p_o - p_i)\, Q\left(\frac{r}{a}, \nu, \frac{b}{a}, \frac{\ell}{a}\right), \tag{143}$$

$$\tau_{rr}(r) = \tau_{rr}^{(0)}(r) + (p_o - p_i) T_{rr}\left(\frac{r}{a}, \nu, \frac{b}{a}, \frac{\ell}{a}\right), \tag{144}$$

$$\tau_{\theta\theta}(r) = \tau_{\theta\theta}^{(0)}(r) + (p_o - p_i) T_{\theta\theta}\left(\frac{r}{a}, \nu, \frac{b}{a}, \frac{\ell}{a}\right), \tag{145}$$

$$u_r(r) = u_r^{(0)}(r) + \frac{p_o - p_i}{G/a} U\left(\frac{r}{a}, \nu, \frac{b}{a}, \frac{\ell}{a}\right), \tag{146}$$

where $(Q, T_{rr}, T_{\theta,\theta}, U)$ are dimensionless functions defined in Appendix B.

It is interesting to note that, whereas the classical solution depends on the individual values of p_i and p_o though A and B, the "gradient correction" is proportional to the difference $p_o - p_i$ (see (143)–(146)). An explanation of this is given at the end of Sect. 4, where is noted that, if $\mathbf{R} = \mathbf{0}$, the "gradient correction" is proportional to the magnitude of $\boldsymbol{\mu}^{(0)}$. In the present problem $\mathbf{R} = \mathbf{0}$, and (128) and (130) show that

$$\boldsymbol{\mu}^{(0)} = \ell^2 \nabla \boldsymbol{\tau}^{(0)} \sim B \sim p_0 - p_i. \tag{147}$$

Therefore, the "gradient correction" is proportional to $p_0 - p_i$, i.e.,

$$\left(f^{(1)}, \mathbf{u}^{(1)}, \boldsymbol{\tau}^{(1)}\right) \sim p_0 - p_i. \tag{148}$$

Figure 3 shows the radial variation of the normalized radial displacement u_r. In Fig. 3 and in all following figures of this section $\nu = 0.3$, $\ell/a = 0.3$, $b/a = 2$, and $p_o/p_i = 2$.

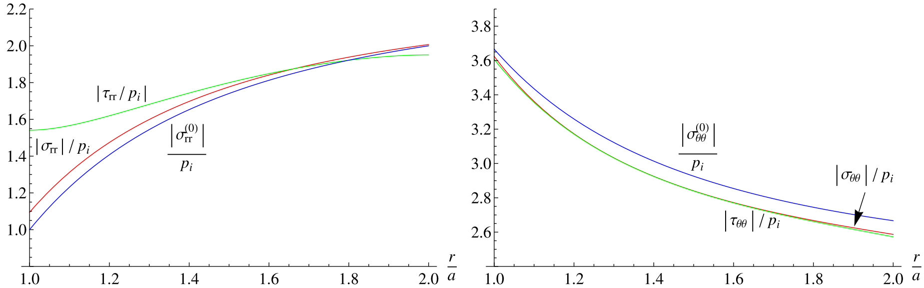


Fig. 4 Spatial variation of normalized radial and hoop stresses for $\nu = 0.3$, $\ell/a = 0.2$, $b = 2a$, and $p_o = 2p_i$. All stress components shown in the figure are compressive

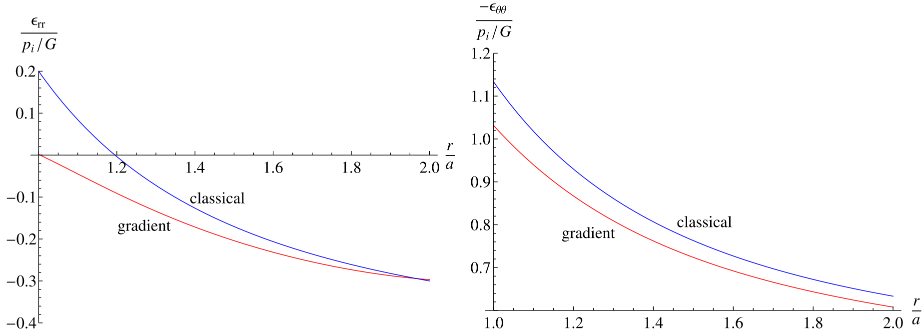


Fig. 5 Spatial variation of normalized strains $\varepsilon_{rr}/(p_i/G)$ and $|\varepsilon_{\theta\theta}|/(p_i/G)$ for $\nu = 0.3$, $\ell/a = 0.2$, $b = 2a$, and $p_o = 2p_i$

As shown in Fig. 3 $|u_r^{\mathrm{gr}}(r)| < |u_r^{\mathrm{cl}}(r)|\ \forall r$, where the superscripts "gr" and "cl" denote the gradient and classical solution respectively; i.e., the gradient elastic material appears to be stiffer. Also, $|u_r^{\mathrm{gr}}(r)|$ varies monotonically with r, whereas the classical solution is such that $|u_r^{\mathrm{cl}}(r)|$ has a minimum at $r \simeq 1.2a$.

Figure 4 shows the spatial variation of the normalized radial and hoop stress components of $\boldsymbol{\tau}$, the true stress $\boldsymbol{\sigma}$, and the classical solution $\boldsymbol{\sigma}^{(0)}$. All stress components shown in Fig. 4 are compressive. The gradient solution predicts a higher compressive radial stress and a lower compressive hoop stress. It is interesting to note that τ_{rr} is substantially different from the true stress σ_{rr}, whereas $\tau_{\theta\theta} \simeq \sigma_{\theta\theta}$.

Figure 5 shows the spatial variation of the normalized strain components $\varepsilon_{rr}/(p_i/G)$ and $|\varepsilon_{\theta\theta}|/(p_i/G)$ for both the classical and the gradient solution. The radial strain ε_{rr} in the gradient solution remains compressive everywhere in the annulus, whereas both tensile and compressive radial strains appear in the classical solution, reflecting the fact that $|u_r^{\mathrm{cl}}(r)|$ has a minimum in the range $a \le r \le b$ (see Fig. 3).

We consider next the case of a thin-walled annulus. Let t be the thickness and R the mean radius of the annulus, i.e.,

$$t = b - a, \qquad R = \frac{a+b}{2}, \qquad \text{so that } a = R - \frac{t}{2},\ b = R + \frac{t}{2}, \tag{149}$$

where

$$\frac{t}{R} \equiv \epsilon \ll 1. \tag{150}$$

If we set $a = R - t/2$, $b = R + t/2$, evaluate (133), (134), (139), and (140) at $r = R$, and expand the solution in $\epsilon = t/R$, we find that

$$\tau_{rr}(R) = -(p_o - p_i)\,\frac{\frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}{1 + 2\frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}\,\frac{R}{t} - \frac{p_o + p_i}{2} + O\left(\epsilon p\right), \tag{151}$$

$$\tau_{\theta\theta}(R) = -(p_o - p_i)\,\frac{1 + \frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}{1 + 2\frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}\,\frac{R}{t} - \frac{p_o + p_i}{2} + O\left(\epsilon p\right), \tag{152}$$

and

$$\sigma_{rr}(R) = -(p_o - p_i)\,\frac{\frac{3-2\nu}{3(1-\nu)}\left(\frac{\ell}{R}\right)^2}{1 + 2\frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}\,\frac{R}{t} - \frac{p_o + p_i}{2} + O\left(\epsilon p\right), \tag{153}$$

$$\sigma_{\theta\theta}(R) = -(p_o - p_i)\,\frac{1 + \frac{(3-2\nu)(4-5\nu)}{3(1-\nu)^2}\left(\frac{\ell}{R}\right)^2}{1 + 2\frac{3-2\nu}{1-\nu}\left(\frac{\ell}{R}\right)^2}\,\frac{R}{t} - \frac{p_o + p_i}{2} + O\left(\epsilon p\right), \tag{154}$$

where p is a typical pressure of order p_o or p_i.

The corresponding thin-wall solution of the classical theory ($\ell = 0$) is

$$\sigma_{rr}^{(0)}(R) = \tau_{rr}^{(0)} = -\frac{p_o + p_i}{2} + O\left(\epsilon p\right), \tag{155}$$

$$\sigma_{\theta\theta}^{(0)}(R) = \tau_{\theta\theta}^{(0)} = -(p_o - p_i)\,\frac{R}{t} - \frac{p_o + p_i}{2} + O\left(\epsilon p\right). \tag{156}$$

It is interesting to note that the gradient theory compared to the classical local theory predicts higher values for the radial stress and lower values for the hoop stress. In particular, whereas the classical theory predicts a radial stress of order p, the gradient theory predicts a radial stress of order $\frac{p}{\epsilon}\left(\frac{\ell}{R}\right)^2$.

7.2 Infinite Body with Cylindrical Hole

We consider the problem of a infinite body with a cylindrical hole of radius a (Fig. 6). The hole is loaded by an internal pressure p_i and a pressure p_o is applied at infinity (Fig. 6). The solution to this problem can be found from the solution developed in Sect. 7.1 by considering the limit $b \to \infty$.

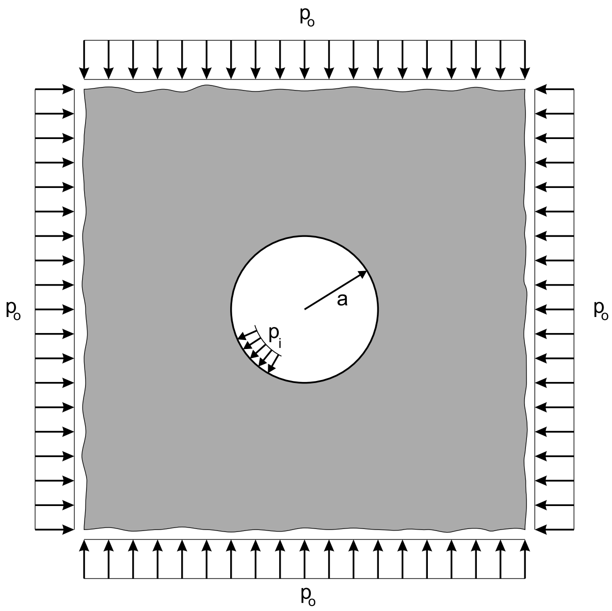
In the limit as $b \to \infty$, the constants that enter the solution take the values

$$A = -p_o, \qquad B = (p_o - p_i)a^2, \qquad c_3 = c_7 = 0, \tag{157}$$

and

$$c_2 = -\frac{p_o - p_i}{c}, \qquad c_8 = 2\frac{p_o - p_i}{c}a\ell K_1\left(\frac{a}{\ell}\right), \tag{158}$$

Fig. 6 Infinite body with circular hole



with

$$c = \frac{1-2\nu}{2} K_0\left(\frac{a}{\ell}\right) + \frac{1-\nu}{2}\left(\frac{a}{\ell} + 4\frac{\ell}{a}\right) K_1\left(\frac{a}{\ell}\right) \tag{159}$$

and the solution becomes

$$\tau_{rr}(r) = -p_o + (p_o - p_i)\frac{a^2}{r^2} - \frac{p_o - p_i}{c}\left\{\frac{a\ell}{r^2} K_1\left(\frac{a}{\ell}\right) + (1-\nu) K_0\left(\frac{r}{\ell}\right) + (1-2\nu)\frac{\ell}{r} K_1\left(\frac{r}{\ell}\right)\right\}, \tag{160}$$

$$\tau_{\theta\theta}(r) = -p_o - (p_o - p_i)\frac{a^2}{r^2} + \frac{p_o - p_i}{c}\left[\frac{a\ell}{r^2} K_1\left(\frac{a}{\ell}\right) - \nu K_0\left(\frac{r}{\ell}\right) + (1-2\nu)\frac{\ell}{r} K_1\left(\frac{r}{\ell}\right)\right], \tag{161}$$

$$f(r) = -\frac{1}{2} p_o r^2 + (p_o - p_i) a^2 \ln r - \frac{p_o - p_i}{c} a\ell K_1\left(\frac{a}{\ell}\right) \ln r. \tag{162}$$

The corresponding displacement field is

$$u_r(r) = -(1-2\nu)\frac{p_o}{2G} r - \frac{p_o - p_i}{2G}\frac{a^2}{r} + \frac{p_o - p_i}{2Gc}\ell\left[\frac{a}{r} K_1\left(\frac{a}{\ell}\right) + (1-2\nu) K_1\left(\frac{r}{\ell}\right)\right]. \tag{163}$$

This solution was communicated to the author by Prof. Exadaktylos [15] in 2001.

The corresponding non-zero in-plane true stresses and true couple-stresses are

$$\sigma_{rr} = -p_0 + (p_0 - p_i)\frac{a^2}{r^2} - \frac{p_0 - p_i}{c}\left[\frac{a\ell}{r^2}K_1\left(\frac{r}{\ell}\right) - \frac{2\nu}{3}\frac{\ell}{r}K_1\left(\frac{r}{\ell}\right)\right], \tag{164}$$

$$\sigma_{\theta\theta} = -p_0 - (p_0 - p_i)\frac{a^2}{r^2} + \frac{p_0 - p_i}{c}\left\{\frac{a\ell}{r^2}K_1\left(\frac{a}{\ell}\right) - \frac{2\nu}{3}\left[K_0\left(\frac{r}{\ell}\right) + \frac{\ell}{r}K_1\left(\frac{r}{\ell}\right)\right]\right\}, \tag{165}$$

and

$$\bar{\bar{\mu}}_{3\theta} = -\bar{\bar{\mu}}_{\theta 3} = -\frac{2\nu}{3}\frac{p_0 - p_i}{c}\ell K_1\left(\frac{r}{\ell}\right). \tag{166}$$

Acknowledgements Fruitful discussions with Prof. A.E. Giannakopoulos of the University of Thessaly are gratefully acknowledged. The author would like also to thank Prof. D. Panagiotounakos of the National Technical University of Athens for helpful discussions and material.

Appendix A: Plane Strain Compatibility Equations in Polar Coordinates for Multiply-Connected Regions

In multiply connected domains the additional compatibility conditions are

$$\mathbf{S}^u = \oint_{C_n} [\boldsymbol{\varepsilon} + \mathbf{x} \times (\nabla \times \boldsymbol{\varepsilon})] \cdot d\mathbf{x} = \mathbf{0}, \tag{167}$$

$$\mathbf{S}^\omega = \oint_{C_n} (\nabla \times \boldsymbol{\varepsilon}) \cdot d\mathbf{x} = \mathbf{0}, \tag{168}$$

$$\mathbf{P}^\varepsilon = \oint_{C_n} (d\mathbf{x} \cdot \boldsymbol{\kappa}) = \mathbf{0}. \tag{169}$$

The polar coordinates of the quantities that appear in gradient elasticity theories can be found in Appendix A of Aravas and Giannakopoulos [6]. Here, we present the form of the above compatibility equations in polar coordinates.

We consider the plane-strain problem of Sect. 6 and introduce polar coordinates (r, θ). The position vector $\mathbf{x}$, its differential $d\mathbf{x}$, and the gradient operator on the plane are

$$\mathbf{x} = r\mathbf{e}_r, \quad d\mathbf{x} = dr\,\mathbf{e}_r + r\,d\theta\,\mathbf{e}_\theta, \quad \nabla = \frac{\partial}{r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\mathbf{e}_\theta, \tag{170}$$

where $(\mathbf{e}_r, \mathbf{e}_\theta)$ are the unit vectors of the polar coordinate system.

The strain tensor $\boldsymbol{\varepsilon}$ and the strain gradient tensor $\boldsymbol{\kappa}$ can be written as

$$\boldsymbol{\varepsilon} = \varepsilon_{rr}\,\mathbf{e}_r\,\mathbf{e}_r + \varepsilon_{\theta\theta}\,\mathbf{e}_\theta\,\mathbf{e}_\theta + \varepsilon_{r\theta}\,(\mathbf{e}_r\,\mathbf{e}_\theta + \mathbf{e}_\theta\,\mathbf{e}_r), \tag{171}$$

$$\begin{aligned}\boldsymbol{\kappa} = {} & \kappa_{rrr}\,\mathbf{e}_r\,\mathbf{e}_r\,\mathbf{e}_r + \kappa_{r\theta\theta}\,\mathbf{e}_r\,\mathbf{e}_\theta\,\mathbf{e}_\theta + \kappa_{rr\theta}\,\mathbf{e}_r\,(\mathbf{e}_r\,\mathbf{e}_\theta + \mathbf{e}_\theta\,\mathbf{e}_r) \\ & + \kappa_{\theta rr}\,\mathbf{e}_\theta\,\mathbf{e}_r\,\mathbf{e}_r + \kappa_{\theta\theta\theta}\,\mathbf{e}_\theta\,\mathbf{e}_\theta\,\mathbf{e}_\theta + \kappa_{\theta r\theta}\,\mathbf{e}_\theta\,(\mathbf{e}_r\,\mathbf{e}_\theta + \mathbf{e}_\theta\,\mathbf{e}_r).\end{aligned} \tag{172}$$

We can evaluate the quantities that appear in (167)–(169) as follows:

$$\begin{aligned}
&[\boldsymbol{\varepsilon} + \mathbf{x} \times (\nabla \times \boldsymbol{\varepsilon})] \cdot d\mathbf{x} \\
&= \left[\varepsilon_{rr}\, \mathbf{e}_r - \left(r \frac{\partial \varepsilon_{r\theta}}{\partial r} - \frac{\partial \varepsilon_{rr}}{\partial \theta} + \varepsilon_{r\theta}\right) \mathbf{e}_\theta\right] dr \\
&\quad + r\left[\varepsilon_{r\theta}\mathbf{e}_r - \left(r \frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\partial \varepsilon_{r\theta}}{\partial \theta} + 2\varepsilon_{\theta\theta} - \varepsilon_{rr}\right) \mathbf{e}_\theta\right] d\theta,
\end{aligned} \tag{173}$$

$$\begin{aligned}
&(\nabla \times \boldsymbol{\varepsilon}) \cdot d\mathbf{x} \\
&= \left[\left(\frac{\partial \varepsilon_{r\theta}}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_{rr}}{\partial \theta} + \frac{2\varepsilon_{r\theta}}{r}\right) dr + \left(r \frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\partial \varepsilon_{r\theta}}{\partial \theta} + \varepsilon_{\theta\theta} - \varepsilon_{rr}\right) d\theta\right] \mathbf{e}_3,
\end{aligned} \tag{174}$$

$$\begin{aligned}
d\mathbf{x} \cdot \boldsymbol{\kappa} &= [\kappa_{rrr}\mathbf{e}_r\mathbf{e}_r + \kappa_{r\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + \kappa_{rr\theta}(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r)]\, dr \\
&\quad + [\kappa_{\theta rr}\mathbf{e}_r\mathbf{e}_r + \kappa_{\theta\theta\theta}\mathbf{e}_\theta\mathbf{e}_\theta + \kappa_{\theta r\theta}(\mathbf{e}_r\mathbf{e}_\theta + \mathbf{e}_\theta\mathbf{e}_r)]\, r\, d\theta.
\end{aligned} \tag{175}$$

In the above equations the unit vectors $(\mathbf{e}_r, \mathbf{e}_\theta)$ are functions of θ:

$$\mathbf{e}_r(\theta) = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2, \qquad \mathbf{e}_\theta(\theta) = -\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2, \tag{176}$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ are the base vectors of a fixed Cartesian coordinate system in the plane where the polar system is defined.

On a circular contour of radius r, $dr = 0$ and the conditions (167)–(169) can be written as follows:

$$\mathbf{S}^u = r\int_0^{2\pi} \left[\varepsilon_{r\theta}\, \mathbf{e}_r - \left(r \frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\partial \varepsilon_{r\theta}}{\partial \theta} + 2\varepsilon_{\theta\theta} - \varepsilon_{rr}\right) \mathbf{e}_\theta\right] d\theta = \mathbf{0}, \tag{177}$$

$$\mathbf{S}^\omega = r\int_0^{2\pi} \left(\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_{r\theta}}{\partial \theta} - \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r}\right) d\theta\, \mathbf{e}_3 = \mathbf{0}, \tag{178}$$

$$\mathbf{P}^\varepsilon = r\int_0^{2\pi} [\kappa_{\theta rr}\, \mathbf{e}_r\, \mathbf{e}_r + \kappa_{\theta\theta\theta}\, \mathbf{e}_\theta\, \mathbf{e}_\theta + \kappa_{\theta r\theta}\, (\mathbf{e}_r\, \mathbf{e}_\theta + \mathbf{e}_\theta\, \mathbf{e}_r)]\, d\theta = \mathbf{0}, \tag{179}$$

where $\mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta(\theta)$ are defined by (176).

All the above equations can be written in terms of the components of $\boldsymbol{\tau}$ if we use the constitutive equations

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left(\tau_{\alpha\beta} - \nu\tau_{\gamma\gamma}\delta_{\alpha\beta}\right), \qquad \kappa_{\alpha\beta\gamma} = \frac{1}{2G\ell^2}\left[(\nabla\boldsymbol{\tau})_{\alpha\beta\gamma} - \nu\,(\nabla\tau_{\delta\delta})_\alpha\, \delta_{\beta\gamma}\right], \tag{180}$$

where the polar components of $\nabla\boldsymbol{\tau}$ are defined by (96) and

$$\nabla\tau_{\delta\delta} = \frac{\partial(\tau_{rr} + \tau_{\theta\theta})}{\partial r}\, \mathbf{e}_r + \frac{1}{r}\frac{\partial(\tau_{rr} + \tau_{\theta\theta})}{\partial \theta}\, \mathbf{e}_\theta. \tag{181}$$

In axisymmetric problems we have that

$$\varepsilon_{r\theta} = 0, \qquad \kappa_{rr\theta} = \kappa_{r\theta r} = \kappa_{\theta rr} = \kappa_{\theta\theta\theta} = 0, \qquad \frac{\partial}{\partial\theta} = 0. \tag{182}$$

In that case, if we take into account (176) and that the axisymmetric solution is independent of θ, we conclude that (177) and (179) are satisfied identically, i.e.,

$$\mathbf{S}^u = \mathbf{P}^\varepsilon = \mathbf{0}, \tag{183}$$

whereas (178) takes the form

$$\mathbf{S}^\omega = \frac{2\pi}{G} r \left[-\nu \frac{d\tau_{rr}}{dr} + (1-\nu) \frac{d\tau_{\theta\theta}}{dr} - \frac{\tau_{rr} - \tau_{\theta\theta}}{r} \right] \mathbf{e}_3 = \mathbf{0}. \tag{184}$$

Appendix B: Constants in the Solution of the Annulus Problem

The constants c_2, c_3, c_7, and c_8 that enter (132)–(140) of the solution of the annulus problem take the values

$$\frac{c_2}{\frac{2(p_o-p_i)}{\Delta}\frac{a\ell}{b^2}} = \frac{1-\nu}{1-2\nu}\left[\frac{a^3}{b^3} I_1\left(\frac{a}{\ell}\right) - I_1\left(\frac{b}{\ell}\right)\right] - \frac{\ell}{b}\left[\frac{a^2}{b^2} I_2\left(\frac{a}{\ell}\right) - I_2\left(\frac{b}{\ell}\right)\right], \tag{185}$$

$$\frac{c_3}{\frac{2(p_o-p_i)}{\Delta}\frac{a\ell}{b^2}} = \frac{1-\nu}{1-2\nu}\left[\frac{a^3}{b^3} K_1\left(\frac{a}{\ell}\right) - K_1\left(\frac{b}{\ell}\right)\right] + \frac{\ell}{b}\left[\frac{a^2}{b^2} K_2\left(\frac{a}{\ell}\right) - K_2\left(\frac{b}{\ell}\right)\right], \tag{186}$$

$$\begin{aligned}
\frac{c_7}{\frac{4(p_o-p_i)}{\Delta}\frac{a\ell^2}{b(b^2-a^2)}} = {} & \left(1+\frac{a^2}{b^2}\right)\frac{\ell^2}{b^2} - \frac{a}{b}\left\{\frac{\ell}{b}\left[I_2\left(\frac{b}{\ell}\right) K_1\left(\frac{a}{\ell}\right) + \frac{a}{b} I_2\left(\frac{a}{\ell}\right) K_1\left(\frac{b}{\ell}\right)\right]\right. \\
& - \left[\frac{1-\nu}{1-2\nu}\left(1-\frac{a^2}{b^2}\right) K_1\left(\frac{a}{\ell}\right) - \frac{a\ell}{b^2} K_2\left(\frac{a}{\ell}\right)\right] I_1\left(\frac{b}{\ell}\right) \\
& \left. + \left[\frac{1-\nu}{1-2\nu}\left(1-\frac{a^2}{b^2}\right) K_1\left(\frac{b}{\ell}\right) + \frac{\ell}{b} K_2\left(\frac{b}{\ell}\right)\right] I_1\left(\frac{a}{\ell}\right)\right\},
\end{aligned} \tag{187}$$

$$\begin{aligned}
\frac{c_8}{\frac{4(p_o-p_i)}{\Delta}\frac{a^2\ell^2}{b^2-a^2}} = {} & 2\frac{a\ell^2}{b^3} - \frac{\ell}{b}\left[I_2\left(\frac{b}{\ell}\right) K_1\left(\frac{a}{\ell}\right) + \frac{a^3}{b^3} I_2\left(\frac{a}{\ell}\right) K_1\left(\frac{b}{\ell}\right)\right] \\
& + \left[\frac{1-\nu}{1-2\nu}\left(1-\frac{a^4}{b^4}\right) K_1\left(\frac{a}{\ell}\right) - \frac{a^3\ell}{b^4} K_2\left(\frac{a}{\ell}\right)\right] I_1\left(\frac{b}{\ell}\right) \\
& - \left[\frac{1-\nu}{1-2\nu}\left(1-\frac{a^4}{b^4}\right) K_1\left(\frac{b}{\ell}\right) + \frac{\ell}{b} K_2\left(\frac{b}{\ell}\right)\right] I_1\left(\frac{a}{\ell}\right),
\end{aligned} \tag{188}$$

where

$$\begin{aligned}
\Delta = {} & 4\frac{a\ell^4}{b^5} + \frac{a\ell}{b^2}\left[(1-\nu) - \frac{a^2}{b^2}\left(1-\nu+2\frac{\ell^2}{b^2}\right)\right] I_2\left(\frac{a}{\ell}\right) K_1\left(\frac{b}{\ell}\right) \\
& - \frac{\ell}{b}\left\{\left[2\frac{\ell^2}{b^2} + (1-\nu)\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)\right] K_1\left(\frac{a}{\ell}\right)\right. \\
& \left. + (1-2\nu)\frac{a\ell}{b^2}\left(1-\frac{a^2}{b^2}\right) K_2\left(\frac{a}{\ell}\right)\right\} I_2\left(\frac{b}{\ell}\right) \\
& + \left\{\frac{1-\nu}{1-2\nu}\left(1-\frac{a^2}{b^2}\right)\left[2\frac{\ell^2}{b^2} + \frac{a^2}{b^2}\left(1-\nu+2\frac{\ell^2}{b^2}\right)\right] K_1\left(\frac{a}{\ell}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{a\ell}{b^2}\left[1-\nu-\frac{a^2}{b^2}\left(1-\nu+2\frac{\ell^2}{b^2}\right)\right]K_2\left(\frac{a}{\ell}\right)\Bigg\}I_1\left(\frac{b}{\ell}\right)\\
&+(1-2\nu)\,\frac{a\,\ell^2}{b^3}\left(1-\frac{a^2}{b^2}\right)I_2\left(\frac{a}{\ell}\right)K_2\left(\frac{b}{\ell}\right)\\
&-\Bigg\{\frac{1-\nu}{1-2\nu}\left(1-\frac{a^2}{b^2}\right)\left[2\frac{\ell^2}{b^2}+\frac{a^2}{b^2}\left(1-\nu+2\frac{\ell^2}{b^2}\right)\right]K_1\left(\frac{b}{\ell}\right)\\
&+\frac{\ell}{b}\left[2\frac{\ell^2}{b^2}+(1-\nu)\,\frac{a^2}{b^2}\left(1-\frac{a^2}{b^2}\right)\right]K_2\left(\frac{b}{\ell}\right)\Bigg\}I_1\left(\frac{a}{\ell}\right). \qquad (189)
\end{aligned}
$$

In the limiting case of an incompressible material ($\nu \to 1/2$), the constants take the following values

$$
c_2 \to (p_o - p_i)\,\frac{8\frac{\ell}{b}[\frac{b^3}{a^3}I_1(\frac{b}{\ell}) - I_1(\frac{a}{\ell})]}{(\frac{b^2}{a^2}-1)[1+4\frac{\ell^2}{b^2}(\frac{b^2}{a^2}+1)][I_1(\frac{a}{\ell})K_1(\frac{b}{\ell}) - I_1(\frac{b}{\ell})K_1(\frac{a}{\ell})]}, \qquad (190)
$$

$$
c_3 \to (p_o - p_i)\,\frac{4\frac{\ell}{b}[\frac{b^3}{a^3}K_1(\frac{b}{\ell}) - K_1(\frac{a}{\ell})]}{(\frac{b^2}{a^2}-1)[1+4\frac{\ell^2}{b^2}(\frac{b^2}{a^2}+1)][I_1(\frac{a}{\ell})K_1(\frac{b}{\ell}) - I_1(\frac{b}{\ell})K_1(\frac{a}{\ell})]}, \qquad (191)
$$

$$
c_7 \to (p_o - p_i)\,\frac{8\frac{\ell^2 b^2}{a^4}}{(\frac{b^2}{a^2}-1)[\frac{b^2}{a^2}+4\frac{\ell^2}{a^2}(\frac{b^2}{a^2}+1)]}, \qquad (192)
$$

$$
c_8 \to (p_o - p_i)\,\frac{8\,\ell^2\frac{b^2}{a^2}(\frac{b^2}{a^2}+1)}{(\frac{b^2}{a^2}-1)[\frac{b^2}{a^2}+4\frac{\ell^2}{a^2}(\frac{b^2}{a^2}+1)]}. \qquad (193)
$$

References

1. Aifantis, E.C.: On the microstructural origin of certain inelastic models. J. Eng. Mater. Technol. **106**, 326–330 (1984)
2. Aifantis, E.C.: On the role of gradients in the localization of deformation and fracture. Int. J. Eng. Sci. **30**, 1279–1299 (1992)
3. Aifantis, E.C.: Exploring the applicability of gradient elasticity to certain micro/nano reliability problems. Microsyst. Technol. **15**, 109–115 (2009)
4. Altan, S.B., Aifantis, E.C.: On the structure of mode III crack-tip in gradient elasticity. Scr. Metall. Mater. **26**, 319–324 (1992)
5. Airy, G.B.: On the strains in the interior of beams. Philos. Trans. R. Soc. Lond. **53**, 49–80 (1863)
6. Aravas, N., Giannakopoulos, A.E.: Plane asymptotic crack-tip solutions in gradient elasticity. Int. J. Solids Struct. **46**, 4478–4503 (2009)
7. Boley, B.A., Weiner, J.H.: Theory of Thermal Stresses. Dover, New York (1997) (originally published in 1960 by John Wiley and Sons, Inc.)
8. Boresi, A.P., Chong, K.P.: Elasticity in Engineering Mechanics, 2nd edn. Wiley, New York (2000)
9. de Borst, R.: Simulation of strain localisation: A reappraisal of the Cosserat continuum. Eng. Comput. **8**, 317–332 (1991)
10. Carlson, D.E.: Stress functions for plane problems with couple stresses. Z. Angew. Math. Phys. **17**, 789–792 (1966)
11. Carlson, D.E.: Stress functions for couple and dipolar stresses. Q. Appl. Math. **24**, 29–35 (1966)
12. Carlson, D.E.: On Günther's stress functions for couple stresses. Q. Appl. Math. **25**, 139–146 (1967)
13. Carlson, D.E.: On general solution of stress equations of equilibrium for a Cosserat continuum. J. Appl. Mech. **34**, 245–246 (1967)
14. Courant, R., John, F.: Introduction to Calculus and Analysis, vol. II. Springer, Berlin (1999) (originally published in 1974 by Interscience Publishers, John Wiley and Sons, Inc.)

15. Exadaktylos, G.: On the problem of a circular hole in an elastic material with microstructure. Private communication (2001)
16. Fleck, N.A., Hutchinson, J.W.: A phenomenological theory for strain gradient effects in plasticity. J. Mech. Phys. Solids **41**, 1825–1857 (1993)
17. Fleck, N.A., Hutchinson, J.W.: Strain gradient plasticity. Adv. Appl. Mech. **33**, 295–361 (1997)
18. Fleck, N.A., Muller, G.M., Ashby, M.F., Hutchinson, J.W.: Strain gradient plasticity: theory and experiment. Acta Metall. Mater. **42**, 475–487 (1994)
19. Fosdick, R., Royer-Carfagni, G.: A Stokes theorem for second-order tensor fields and its implications in continuum mechanics. Int. J. Non-Linear Mech. **40**, 381–386 (2005)
20. Fraeijs de Veubeke, B.M.: A Course in Elasticity. Springer, Berlin (1979)
21. Fung, Y.C.: Foundations of Solid Mechanics. Prentice Hall, New York (1965)
22. Georgiadis, H.G., Vardoulakis, I., Velgaki, E.G.: Dispersive Rayleigh-wave propagation in microstructured solids characterized by dipolar gradient elasticity. J. Elast. **74**, 17–45 (2004)
23. Germain, P.: Sur l'application de la méthode des puissances virtuelles en mécanique des milieux continus. C. R. Acad. Sci. Paris **274**, 1051–1055 (1972)
24. Germain, P.: The method of virtual power in Continuum Mechanics. Part 2: Microstructure. SIAM J. Appl. Math. **25**, 556–575 (1973)
25. Germain, P.: La méthode des puissances virtuelles en mécanique des milieux continus. J. Méc. **12**, 235–274 (1973)
26. Günther, W.: Zur Statik und Kinematik des Cosseratschen Kontinuums. Abh. Braunschw. Wiss. Ges. **70**, 195–213 (1958)
27. Koiter, W.T.: Couple-stresses in the theory of elasticity. I. Proc. K. Ned. Akad. Wet., Ser. B Phys. Sci. **67**, 17–29 (1964)
28. Koiter, W.T.: Couple-stresses in the theory of elasticity. II. Proc. K. Ned. Akad. Wet., Ser. B Phys. Sci. **67**, 30–44 (1964)
29. Lazar, M., Maugin, G.A.: Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. Int. J. Eng. Sci. **43**, 1157–1184 (2005)
30. Lazar, M., Maugin, G.A.: A note on line forces in gradient elasticity. Mech. Res. Commun. **33**, 674–680 (2006)
31. Lazar, M., Maugin, G.A.: Dislocations in gradient elasticity revisited. Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci. **462**, 3465–3480 (2006)
32. Leblond, J.B., Perrin, G., Devaux, J.: Bifurcation effects in ductile materials with damage localization. J. Appl. Mech. **61**, 236–242 (1994)
33. Mindlin, R.D.: Influence of couple-stresses on stress concentrations. Exp. Mech. **3**, 1–7 (1963)
34. Mindlin, R.D.: Microstructure in linear elasticity. Arch. Ration. Mech. Anal. **10**, 51–78 (1964)
35. Mindlin, R.D.: Second gradient of strain and surface tension in linear elasticity. Int. J. Solids Struct. **1**, 417–438 (1965)
36. Mindlin, R.D., Eshel, N.N.: On first strain-gradient theories in linear elasticity. Int. J. Solids Struct. **4**, 109–124 (1968)
37. Mindlin, R.D., Tiersten, H.F.: Effects of couple-stresses in linear elasticity. Arch. Ration. Mech. Anal. **11**, 415–448 (1962)
38. Nowacki, W.: Theory of Asymmetric Elasticity. Pergamon Press, Elmsford (1985)
39. Pijaudier-Cabot, G., Bazant, Z.P.: Nonlocal damage theory. J. Eng. Mech. **113**, 1512–1533 (1987)
40. Schaefer, H.: Versuch einer Elastizitätstheorie des zweidimensionalen ebenen Cosserat-Kontinuums. In: Schäfer, M. (ed.) Miszellaneen der Angewandten Mechanik (Festschrift Walter Tollmien zum 60. Geburtstag am 13 Oktober 1960), pp. 277–292. Akademie-Verlag, Berlin (1962)
41. Soutas-Little, R.W.: Elasticity. Dover, New York (1999) (originally published in 1973 by Prentice Hall, Inc.)
42. Toupin, R.A.: Elastic materials with couple-stresses. Arch. Ration. Mech. Anal. **11**, 385–414 (1962)
43. Tvergaard, V., Needleman, A.: Effects of nonlocal damage in porous plastic solids. Int. J. Solids Struct. **32**, 1063–1077 (1995)
44. Vardoulakis, I.: Shear-banding and liquefaction in granular materials on the basis of a Cosserat continuum theory. Arch. Appl. Mech. **59**, 106–113 (1989)
45. Vardoulakis, I., Aifantis, E.C.: Gradient dependent dilatancy and its implications in shear banding and liquefaction. Arch. Appl. Mech. **59**, 197–208 (1989)
46. Vardoulakis, I., Sulem, J.: Bifurcation Analysis in Geomechanics. Blackie Academic and Professional (Chapman Hall), Glasgow (1995)