# The friction factor of a rarefied gas flow in a circular tube

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An expression for the estimation of the Poiseuille number for internal rarefied flows is developed. The derived formula is given in terms of dimensionless quantities, which are obtained by the solution of the flow based on kinetic theory. The formulation is applied to the fully developed flow of a gas in a circular tube, and tabulated results of the Poiseuille number are presented in the whole range of the Knudsen number and for various values of the accommodation coefficient. Also, in the slip regime, a closed-form expression for the Poiseuille number is derived and a brief discussion on the proper estimation of the viscous slip coefficient is provided. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2780192]

In viscous flows one of the most meaningful parameters characterizing the flow is the friction factor. On many occasions, including the case of internal fully developed laminar flows, which is investigated in the present work, instead of the friction factor it is more appropriate to estimate the Poiseuille number Po of the flow, which is defined as

$$\mathrm{Po} = \frac{8\overline{\tau_w'}D_h}{\mu\overline{u'}}. \tag{1}$$

Here, $\bar{\tau}_w'$ is the mean wall shear stress, $\mu$ is the viscosity, $D_h$ is the hydraulic diameter, and $\bar{u}$ is the mean velocity. The mean quantities $\bar{\tau}_w'$ and $\bar{u}$ are computed by integrating accordingly the corresponding distributions over the perimeter and the area of the flow cross section. It may be useful to note that in several references,[1,2] the Poiseuille number is given as

$$\mathrm{Po} = f\mathrm{Re}, \tag{2}$$

where $f$ is the Darcy friction factor and Re is the Reynolds number of the flow based on the hydraulic diameter. Equations (1) and (2) from a mathematical point of view are identical. However, from a physical point of view, Eq. (1) is the correct one in the case of laminar flows, since the shear stress is scaled by the viscous stress and not by dynamic pressure, as it is done in Eq. (2), which is more appropriate in the case of turbulent flows.

Over the years, the Poiseuille number of an extensive number of flow configurations has been determined. Most of this work is based on the solution of the classical hydrodynamic equations subject to no-slip boundary conditions and therefore it is valid only at the continuum limit. Recently, the Poiseuille number of certain fully developed flows in the slip regime (not far from the continuum limit) has been also estimated, based on the hydrodynamic equations with slip boundary conditions.[2–6]

However, it is well known that when the flow is in the transition regime or at the free molecular regime (medium and highly rarefied flows) the hydrodynamic equations are not valid and a kinetic approach based on the Boltzmann equation or reliable kinetic models is required.[7,8] The main advantage of a kinetic solution is the fact that the results are valid in the whole range of the Knudsen number from the free molecular through the transition and slip regimes up to the hydrodynamic limit. It is interesting to note that although kinetic solutions are available for a relative large number of internal rarefied flows,[9–11] no estimates for the Poiseuille number of such flows have been reported.

In the present work following a simple procedure we provide an expression of the Poiseuille number in terms of nondimensional quantities obtained by kinetic solutions. This expression is general and can be applied to internal rarefied gas flows of any cross section. Then, this analysis is applied to the flow of a gas through a circular tube and tabulated results of the Poiseuille number are presented in the whole range of gas rarefaction and for various values of the accommodation coefficient. At the hydrodynamic limit the well-known result, Po=64, is recovered.

Consider the flow of a rarefied gas through a channel with a cross-section area $A'$ and perimeter $\Gamma'$. The basic flow parameter is the Knudsen number or alternatively the rarefaction parameter $\delta$, which is commonly used in the rarefied gas community. They are defined as

$$\delta = \frac{P_0 D_h}{\mu v_0} = \frac{\sqrt{\pi}}{2}\frac{D_h}{\lambda} = \frac{\sqrt{\pi}}{2}\frac{1}{\mathrm{Kn}}. \tag{3}$$

Here, $P_0$ is a reference pressure, $D_h = 4A'/\Gamma'$ is the hydraulic diameter, $\mu$ is the gas viscosity at reference temperature $T_0$, and $v_0 = \sqrt{2RT_0}$ is the characteristic molecular velocity, with $R = k/m$ denoting the gas constant ($k$ is the Boltzmann constant and $m$ the molecular mass). The rarefaction parameter $\delta$ is proportional to the inverse Knudsen number Kn, which is defined as the ratio of the mean free path $\lambda$ over $D_h$. The hydraulic diameter $D_h$ and the molecular velocity $v_0$ are taken as the characteristic length and velocity of the problem. The dimensionless area and perimeter are $A = A'/D_h^2$ and $\Gamma = \Gamma'/D_h$, respectively, and it is readily seen that the ratio $A/\Gamma = 1/4$.

In kinetic solutions, the dimensionless pressure gradient driving the flow is[12,13]

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$$X_P = \frac{D_h}{P_0}\frac{dP'}{dz'} = \frac{1}{P_0}\frac{dP'}{dz}, \tag{4}$$

where the spatial variable $z=z'/D_h$ is the flow direction. Also, the dimensionless mean macroscopic velocity and wall shear stress are defined as

$$\bar{u} = \frac{\overline{u'}}{\upsilon_0 X_P} \tag{5}$$

and

$$\overline{\tau_w} = \frac{\overline{\tau_w'}}{2P_0 X_P}, \tag{6}$$

respectively. To obtain the dimensionless quantities $\bar{u}$ and $\overline{\tau_w}$ via kinetic theory, a kinetic equation is first solved numerically for the unknown distribution function and then the corresponding moments of the distribution function are estimated.

Since the flow is fully developed and there is no net momentum flux in the $z'$ direction, the net pressure and the wall shear stress are equated to yield

$$\overline{\tau_w'} = \frac{A'}{\Gamma'}\frac{dP'}{dz'}. \tag{7}$$

By nondimensionalizing Eq. (7) it is easily deduced that

$$\overline{\tau_w} = \frac{A}{2\Gamma} = \frac{1}{8}. \tag{8}$$

This result is always valid independent of the dimensional cross section $A'$ and perimeter $\Gamma'$.

Finally, substituting Eqs. (5) and (6) into Eq. (1) and using, into the resulting equation, Eq. (8), as well as the definition of $\delta$, given by Eq. (3), the following expression for the Poiseuille number is deduced:

$$\mathrm{Po} = \frac{2\delta}{\bar{u}}. \tag{9}$$

Therefore, once the kinetic solution of a rarefied gas flow problem is obtained, the dimensionless mean velocity $\bar{u}$ can be computed, and then Eq. (9) is implemented to estimate the Poiseuille number of the flow. This result is general and may be applied to fully developed gas flows through ducts of various cross sections in the whole range of $\delta$.

Next, we study the specific case of laminar fully developed flow through a circular tube, known as the Hagen–Poiseuille flow. This problem has been solved using kinetic theory by several authors. For completeness purposes and in order to be consistent with the introduced nondimensional analysis we present the formulation of the problem. The rarefaction parameter $\delta$ is defined according to Eq. (3), where $D_h=D$ is the diameter of the tube. Then, since the flow is pressure driven, it may be modeled by the reduced Bhatnagar-Gross-Krook (BGK) kinetic equation, given by[9,14,15]

$$\zeta\left[\cos\theta\frac{\partial Y}{\partial r} - \frac{\sin\theta}{r}\frac{\partial Y}{\partial\theta}\right] + \delta Y = \delta u - \frac{1}{2}. \tag{10}$$

Here, $Y=Y(r,\zeta,\theta)$ is the unknown distribution function, $0 \leq r \leq 1/2$ is the dimensionless spatial variable, while $0 \leq \zeta < \infty$ and $0 \leq \theta \leq 2\pi$ are the magnitude and the polar angle of the dimensionless molecular velocity vector. In addition, in the right-hand side of Eq. (10),

$$u(r) = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} Y\zeta e^{-\zeta^2} d\zeta d\theta, \tag{11}$$

is the dimensionless velocity distribution. The gas surface interaction is modeled by the Maxwell diffuse-specular boundary conditions given by

$$Y^{(+)} = (1-\alpha)Y^{(-)} \text{ at } r = 1/2, \tag{12}$$

where the superscripts (+) and (−) denote distributions leaving from and arriving to the boundary. The parameter $0 < \alpha \leq 1$ is the momentum accommodation coefficient ($\alpha=1$ corresponds to purely diffuse scattering). The flow problem defined by the linear integrodifferential equations (10) and (11), subject to boundary condition (12), is solved numerically and the dimensionless velocity distribution $u(r)$, for each specified $\delta$ and $\alpha$, is obtained. Then, the corresponding dimensionless mean velocity is obtained by integrating $u(r)$ according to

$$\bar{u} = 8\int_0^{1/2} u(r) r dr. \tag{13}$$

Finally, the Poiseuille number is estimated by substituting the corresponding $\bar{u}$ and $\delta$ into Eq. (9).

Based on the above, tabulated results for the Poiseuille number in the whole range of the rarefaction parameter $\delta$ and for $\alpha=1$, 0.85, and 0.7, are presented in Table I. The results may be considered as accurate to all three significant figures provided within ±1 to the last one. In a log−log scale the following remarks can be made. For $10^{-3} \leq \delta \leq 1$, the Po number is increased roughly proportionally to $\delta$. Then, for $10^{-1} < \delta < 10^2$, the Po number keeps increasing as $\delta$ is increased, but in a slower pace. Finally, for $\delta \geq 10^2$, as $\delta$ is increased, the Po number is increased very slowly and it is clear that it is reaching asymptotically the analytic result Po=64 at the continuum limit ($\delta \to \infty$). These remarks apply to all three values of the accommodation coefficient. Also, as $\alpha$ is decreased (more specular reflection) the values of the Po number for the same $\delta$ are decreased.

We continue our discussion on the circular tube problem by deriving analytic expressions for the Poiseuille number in the slip regime. When the Hagen–Poiseuille flow is not far from local equilibrium the dimensionless velocity distribution may be obtained by solving the Poisson equation,[9]

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = -\delta, \quad 0 \leq r < 1/2, \tag{14}$$

subject to the slip boundary condition,

TABLE I. The Poiseuille number Po for rarefied gas flow in a circular tube in the whole range of the rarefaction parameter $\delta$ and for various accommodation coefficients $\alpha$.

| | Po | | |
|---|---|---|---|
| $\delta$ | $\alpha=1.0$ | $\alpha=0.85$ | $\alpha=0.70$ |
| 0.001 | 0.005 33 | 0.003 94 | 0.002 88 |
| 0.01 | 0.0538 | 0.0401 | 0.0294 |
| 0.1 | 0.559 | 0.425 | 0.319 |
| 0.3 | 1.73 | 1.34 | 1.02 |
| 0.5 | 2.90 | 2.28 | 1.75 |
| 1 | 5.77 | 4.60 | 3.58 |
| 1.5 | 8.46 | 6.83 | 5.36 |
| 2 | 11.0 | 8.93 | 7.06 |
| 3 | 15.5 | 12.8 | 10.2 |
| 4 | 19.3 | 16.1 | 13.1 |
| 5 | 22.6 | 19.1 | 15.7 |
| 6 | 25.5 | 21.8 | 18.0 |
| 7 | 28.1 | 24.1 | 20.1 |
| 8 | 30.3 | 26.3 | 22.1 |
| 9 | 32.3 | 28.2 | 23.9 |
| 10 | 34.1 | 29.9 | 25.5 |
| 11 | 35.6 | 31.5 | 27.0 |
| 13 | 38.4 | 34.3 | 29.7 |
| 15 | 40.7 | 36.6 | 32.1 |
| 20 | 44.9 | 41.1 | 36.7 |
| 30 | 50.0 | 46.8 | 42.9 |
| 40 | 53.0 | 50.2 | 46.8 |
| 50 | 54.9 | 52.5 | 49.5 |
| 100 | 59.1 | 57.7 | 55.8 |
| 200 | 61.5 | 60.7 | 59.7 |
| 500 | 63.0 | 62.6 | 62.3 |
| $\infty$ | 64.0 | 64.0 | 64.0 |

$$u\left(\frac{1}{2}\right) = -\frac{\sigma_P}{\delta}\frac{du}{dr}\bigg|_{r=1/2}, \tag{15}$$

where $\sigma_P$ is the viscous slip coefficient (VSC), and it is obtained via kinetic theory. An early rough kinetic estimate of VSC has been provided by Maxwell, more than 100 years ago, in the form[16]

$$\sigma_P = \frac{\sqrt{\pi}}{2}\frac{2-\alpha}{\alpha}. \tag{16}$$

Over the years this estimation has been significantly improved by solving the complete half-space kinetic problem, known as the viscous slip or Kramers problem. It has been found that using the BGK equation, with $\alpha=1$ (purely diffuse scattering), we obtain $\sigma_P=1.016$ (Refs. 17 and 18), while the dependency on the accommodation coefficient can be considered by using the expression[9,19]

$$\sigma_P(\alpha) = \frac{2-\alpha}{\alpha}[\sigma_P(1) - 0.1211(1-\alpha)]. \tag{17}$$

The corresponding results of $\sigma_P$, based on the Boltzmann equation or other kinetic model equations, are very close to the ones obtained by the BGK model.[20,21]

Solving Eq. (14), with boundary condition (15) yields

$$u(r) = \frac{\delta}{4}\left(\frac{1}{4} - r^2\right) + \frac{\sigma_P}{4} \tag{18}$$

and

$$\bar{u} = \frac{\delta}{32} + \frac{\sigma_P}{4}. \tag{19}$$

It is obvious that at the right-hand side of Eqs. (18) and (19), the first terms correspond to the hydrodynamic solution and the second ones to the slip correction. Substituting Eq. (19) into Eq. (9) results in

$$f\,\mathrm{Re} = 64 \bigg/ \left(1 + \frac{8\sigma_P}{\delta}\right) = 64 \bigg/ \left(1 + \frac{16\sigma_P \mathrm{Kn}}{\sqrt{\pi}}\right), \tag{20}$$

where the VSC is obtained by Eq. (17), with $\sigma_p(1)=1.016$. The range of applicability and the accuracy of Eq. (20) in terms of $\delta$ may be easily examined by comparing the estimated Poiseuille numbers with the corresponding ones of Table I. At the hydrodynamic limit ($\delta \to \infty$), Eq. (20) is reduced to the well-known result Po=64. In principal, Eq. (20) is valid in the slip regime ($\delta > 10$), but due to its simplicity it may be used, at some extent, in the transition regime to provide rough estimates.

Even more, if in Eq. (20) the VSC is substituted from expression (16), we find

$$f\,\mathrm{Re} = 64 \bigg/ \left[1 + \left(\frac{2-\alpha}{\alpha}\right)\frac{4\sqrt{\pi}}{\delta}\right] = 64 \bigg/ \left[1 + \left(\frac{2-\alpha}{\alpha}\right)8\mathrm{Kn}\right], \tag{21}$$

which for the specific case of $\alpha=1$ yields the well-known result $f\,\mathrm{Re}=64/(1+8\mathrm{Kn})$. Comparing, say, for $\alpha=1$, the results of Eq. (21) with the corresponding kinetic results in Table I, it is found that using Eq. (21) the relative error on the estimation of the Poiseuille number at $\delta=20$, 10, and 5 is about 5.3%, 9.7%, and 17.3%, respectively. The corresponding errors using Eq. (20) are about 1.3% ($\delta=20$), 3.5% ($\delta=10$), and 8.0% ($\delta=5$). In both cases, as $\alpha$ is decreased the relative error is increased.

It is important to note that estimates, based on the old (erroneous) result by Maxwell for the VSC, are widely reported in the literature and they are used even for comparison with experimental work. Expressions, such as Eq. (20), which are based on the correct computation of VSC, provide more accurate approximations in the slip regime, while they are derived in an equally simple manner.

In summary, we have developed compact expressions for the estimation of the friction factor and of the Poiseuille number in terms of kinetic quantities for internal rarefied flows. The proposed procedure and formulas have been applied to the flow of a rarefied gas in a circular tube, and tabulated results of the Poiseuille number have been provided in the whole range of the Knudsen number and for various values of the accommodation coefficient. In addition, a simple closed-form expression for the Poiseuille number in the slip regime has been derived. The proposed methodology may be applied in a straightforward manner to channels of orthogonal, triangular, and trapezoidal cross sections, which are of some interest in several technological fields including nanofluidics, microfluidics, and vacuum technology.

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[1] F. M. White, *Viscous Fluid Flows* (McGraw-Hill, New York, 1974).
[2] G. L. Morini, M. Spiga, and P. Tartarini, "The rarefaction effect on the friction factor of gas flow in microchannels," Superlattices Microstruct. **35**, 587 (2004).
[3] A. Beskok and G. E. Karniadakis, "A model for flows in channels, pipes and ducts at micro and nano scales," Microscale Thermophys. Eng. **3**, 43 (1999).
[4] A. A. Rostami, A. S. Mujumdar, and N. Saniei, "Flow and heat transfer for gas flowing in microchannels: a review," Heat Mass Transfer **38**, 359 (2001).
[5] C. Y. Wang, "Slip flow in a triangular duct—an exact solution," Z. Angew. Math. Mech. **83**, 629 (2003).
[6] X. Zhu, Q. Liao, and M. D. Xin, "Gas flow in microchannel of arbitrary shape in slip flow regime," Nanoscale and Microscale Thermophysical Engineering **10**, 41 (2006).
[7] J. H. Ferziger and H. G. Kaper, *Mathematical Theory of Transport Processes in Gases* (North-Holland, Amsterdam, 1972).
[8] C. Cercignani, *The Boltzmann Equation and its Application* (Springer, New York, 1988).
[9] F. Sharipov and V. Seleznev, "Data on internal rarefied gas flows," J. Phys. Chem. Ref. Data **27**, 657 (1998).
[10] K. Aoki, "Dynamics of rarefied gas flows: Asymptotic and numerical analyses of the Boltzmann equation," AIAA Paper 2001–0874, 39th AIAA Aerospace Science Meeting and Exhibit, Reno, NV (2001).
[11] Y. Sone, *Kinetic Theory and Fluid Mechanics* (Birkhäuser, Boston, 2002).
[12] F. Sharipov and D. Kalempa, "Gaseous mixture flow through a long tube at arbitrary Knudsen number," J. Vac. Sci. Technol. A **20**, 814 (2002).
[13] S. Naris, D. Valougeorgis, F. Sharipov, and D. Kalempa, "Flow of gaseous mixtures through rectangular microchannels driven by pressure, temperature and concentration gradients," Phys. Fluids **17**, 100607 (2005).
[14] D. Valougeorgis and J. R. Thomas, "Exact numerical results for Poiseuille and thermal creep flow in a cylindrical tube," Phys. Fluids **29**, 423 (1986).
[15] F. Sharipov and V. D. Seleznev, "Rarefied gas flow through a long tube at any pressure ratio," J. Vac. Sci. Technol. A **12**, 2933 (1999).
[16] E. H. Kennard, *Kinetic Theory of Gases* (McGraw-Hill, New York, 1938).
[17] S. K. Loyalka and K. A. Hickey, "Velocity slip and defect: Hard sphere gas," Phys. Fluids A **1**, 612 (1989).
[18] L. B. Barichello, M. Camargo, P. Rodrigues, and C. E. Siewert, "Unified solution to classical flow problems based on the BGK model," Z. Angew. Math. Mech. **52**, 517 (2001).
[19] S. K. Loyalka, "Momentum and temperature slip coefficients with arbitrary accommodation coefficients," J. Chem. Phys. **48**, 5432 (1968).
[20] C. E. Siewert and F. Sharipov, "Model equations in rarefied gas dynamics, Viscous slip and thermal slip coefficients," Phys. Fluids **14**, 4123 (2002).
[21] T. Ohwada, Y. Sone, and K. Aoki, "Numerical analysis of the shear and thermal creep flows of a rarefied gas over a plane wall on the basis of the linearized Boltzmann equation for hard-sphere molecules," Phys. Fluids A **1**, 1588 (1989).