On the Calculations of the Stored Energy of Cold Work

When a metal deforms plastically, most of the mechanical energy expended in the deformation process is converted into heat and the remainder is stored in the material. A method for the calculation of the stored energy from an experimentally determined load-displacement curve of an elastic-plastic structure is presented. The method is applied to the problem of simple tension of a polycrystalline metal and a simple technique for the calculation of the stored energy from the uniaxial stress-strain curve is presented.

1 Introduction

When a metal deforms plastically, most of the mechanical energy expended in the deformation process is converted into heat and the remainder is stored in the material. The stored energy is associated with residual stresses generated in the metal after unloading as well as with the creation of lattice imperfections. A summary of the experimental and theoretical developments concerning the stored energy of cold work has been given in two review articles by Titchener and Bever (1958) and Bever et al. (1973). The reported values of the ratio of the stored to the expended energy range from less than 1 to about 90 percent, with the great majority lying under 15 percent.

A method for the calculation of the stored energy from an experimentally determined load-displacement curve of an elastic-plastic structure is developed in this paper. The effects of hardening and initial stresses are discussed and several examples are presented. The method is applied to the problem of simple tension of a polycrystalline metal and a simple technique for the calculation of the stored energy from the uniaxial stress-strain curve of the metal is developed. The results indicate that the stored energy increases with increasing hardening capacity of the metal and that strain hardening and energy storage cease at the same stage of the deformation process. The paper closes with a comparison of the theoretical predictions with experimental observations.

2 Some Preliminary Results

Consider an ideal elastic-plastic body subjected to a system of loads increasing in proportion to one parameter \( Q > 0 \). We assume that there are no initial stresses in the body and that the loads are slowly varying in time so that any dynamic effects can be neglected. After the application of the loads all strain components are assumed to remain small and the standard strain decomposition into elastic and plastic parts is used:

\[
\varepsilon = \varepsilon^e + \varepsilon^p.
\]

Linear elasticity is assumed so that

\[
\sigma = C : \varepsilon^e = \varepsilon^p : C,
\]

where \( \sigma \) is the stress tensor, and \( C \) is the fourth order elasticity tensor with the usual symmetries.

Let \( q \) be a displacement parameter which is work-conjugate to \( Q \), such that the work done by all the applied loads on the body is given by \( \int_0^Q Qdq \). Then, the principles of virtual displacements and virtual forces can be written as

\[
\int_0^Q Q dq = \int_V \int_0^q \sigma : \varepsilon dV,
\]

and

\[
\int_0^Q q dq = \int_V \int_0^q \varepsilon : \sigma dV,
\]

where \( V \) is the volume of the body. Using the elastic plastic decomposition (2.1) we can write the above equation as

\[
\int_0^q Q dq = W^e + W^p,
\]
and
\[ \int_0^Q q \, dQ = W^e + \int_0^\sigma \sigma : \varepsilon^p : d\sigma \, dV, \]  
(2.6)
where\n\[ W^e = \int_0^\sigma \sigma : \varepsilon^e \, dV = \int_0^\sigma \varepsilon : \varepsilon \, dV \]
(2.7)
is the elastic strain energy of the body, and\n\[ W^p = \int_0^\sigma \sigma : \varepsilon^p \, dV \]
(2.8)
is the plastic energy dissipation.

3 Elastic-Perfectly-Plastic Bodies

If an elastic-perfectly-plastic body undergoes nonhomogeneous deformation, the generalized load-displacement curve will be of the form shown in Fig. 1. The body deforms elastically in O-A until plastic yielding first occurs at a load level A; for higher loads, part of the structure deforms plastically and the load-displacement relationship becomes nonlinear (A-B in Fig. 1). Finally, the unloading part B-C is linear provided that reverse plasticity does not take place anywhere in the body.

The limit load corresponds to the value of the load parameter \( Q_{\text{lim}} \) in Fig. 1.

At maximum load (point B in Fig. 1) equation (2.6) can be written as
\[ \int_0^{Q_B} q \, dQ = W_B^\sigma + \int_{\sigma}^{\varepsilon_B} \varepsilon^p : d\sigma \, dV. \]
(3.1)
where \( \sigma_B \) and \( W_B^\sigma \) are the stress field and the elastic strain energy at load \( Q_B \).

Let \( \sigma^* \) and \( \varepsilon^* \) be the stresses and strains in the corresponding ideal linear elastic body at the same load level \( Q_B \). After unloading, if reverse plasticity does not take place, the residual stresses and strains \( \sigma^r \) and \( \varepsilon^r \) in the plastic body are given by
\[ \sigma^r = \sigma_B - \sigma^*, \]
(3.2)
and\n\[ \varepsilon^r = \varepsilon_B - \varepsilon^*. \]
(3.3)
Using the principle of virtual work and taking into account that \( \sigma^r \) is a self-equilibrated stress field and that \( \varepsilon^r \) is a compatible strain field, we can easily show that
\[ W_B^e = \int_{\frac{1}{2} \sigma_B}^{\frac{1}{2} \varepsilon_B} \varepsilon_B : \varepsilon^p \, dV = W^* + W', \]
(3.4)
where
\[ W^* = \int_{\frac{1}{2} \sigma_B}^{\frac{1}{2} \varepsilon_B} \sigma : \varepsilon^p : d\sigma \, dV = \frac{1}{2} Q_B (\sigma_B - q_B) \]
(3.5)
(see Fig. 1), and
\[ W' = \int_{\frac{1}{2} \sigma_B}^{\sigma_B} \varepsilon^p : C^{-1} : \varepsilon \, dV. \]
(3.6)
It should be noted that \( W^* \) is the energy stored in the material after unloading and is associated with the presence of residual stresses in the structure. Equation (3.1) can be now written as
\[ \int_0^{Q_B} q \, dQ = W^* + W' + \int_{\varepsilon_B}^{\sigma_B} \varepsilon^p : d\sigma \, dV. \]
(3.7)
The area of triangle OEF in Fig. 1 is equal to that of triangle CDB and thus equal to \( W^* \). Therefore, equation (3.7) implies that the shaded area in Fig. 1 is equal to the sum
\[ W' + \int_{\varepsilon_B}^{\sigma_B} \varepsilon^p : d\sigma \, dV. \]
(3.8)
We are going to show next that for certain loading histories the integral in the above expression vanishes and that the stored energy of the body can be calculated easily from the load-displacement curve shown in Fig. 1.

(a) Consider first the case of proportional stressing in which \( \sigma \) increases monotonically in such a way that the components of \( \sigma \) remain in constant proportion to one another. In such a case the stress tensor does not rotate in stress space and, once plastic yielding occurs at a material point, the condition \( d\sigma = 0 \) must be satisfied at that point for continuing plastic deformation.

(b) Consider next the case of an elastic perfectly plastic material with associated flow rule. In this case the plastic strain rate is always normal to the yield surface. If the yield surface includes flat regions and if the stress tensor at every material point remains continuously on one of the flats after yielding, then the direction of the plastic strain rate at every material point is constant and normal to the corresponding flat. This, in turn, implies that the plastic strain itself at each material point is also normal to a single flat during plastic deformation and therefore\n\[ \varepsilon^p : d\sigma = 0. \]

For both cases (a) and (b) mentioned above, the integral in (3.8) vanishes and the shaded area in Fig. 1 is equal to the stored energy \( W^* \) of the body. This observation provides a simple method for the calculation of the stored energy from an experimentally determined load-displacement curve provided that either condition (a) or (b) is satisfied.

For arbitrary loading histories, the integral on the right-hand side of (3.7) does not vanish and after integrating by parts we find
\[ \int_{\sigma_B}^{\varepsilon_B} \varepsilon^p : d\sigma \, dV = -\int_{\sigma_B}^{\varepsilon_B} (\sigma_B - \sigma) : d\varepsilon^p \, dV. \]
(3.9)
If the principle of maximum plastic work (Hill, 1950) is satisfied, i.e., \( (\sigma_B - \sigma) : d\varepsilon^p \leq 0 \), then the right-hand side of equation (3.9) is nonpositive and the shaded area in Fig. 1 provides only a lower bound for \( W^* \).

The effects of local hardening and initial stresses as well as several examples including the application of the method to the problem of tension of a polycrystalline metal are presented in the following.

4 The Effects of Hardening and Initial Stresses

Isotropic hardening with associated flow rule is assumed. The generalized load-displacement curve is of the form shown in Fig. 1; however, a limit load does not exist in this case. We
are going to show in the following that under certain circumstances the shaded area in Fig. 1 provides an upper bound to the energy stored in the structure after unloading.

For both loading histories (a) and (b) considered in Section 3 the direction of the plastic strain rate tensor is constant; therefore, \( \epsilon^p : \dot{\sigma} \geq 0 \) at all material points and equation (3.7) implies that

\[
\int_0^{Q_B} q \, dQ \geq W'' + W^*. 
\]  
(4.1)

After unloading \( W^* \) is released and using an argument similar to that presented in Section 3 one can show that the shaded area in Fig. 1 provides an upper bound for the stored energy of the structure. In a strain softening material the direction of inequality (4.1) is reversed and the shaded area in Fig. 1 provides a lower bound to the stored energy.

If initial stresses \( \sigma_i \) are present in the reference (strain-free) configuration, one can easily show that the shaded area in Fig. 1 is equal to

\[
\Delta W'' + \int_V (\sigma_B - \sigma) : \epsilon : dV - \int'_V (\sigma_B - \sigma) : \epsilon : dV \quad \text{where} \\
\Delta W'' = \int_V \frac{1}{2} \sigma' : \epsilon : C^{-1} : \epsilon' \, dV \\
- \int_V \frac{1}{2} \sigma_1 : \epsilon : C^{-1} : \epsilon_1 \, dV
\]

(4.2)

is the increase in stored energy. In that case, the sign of second term in (4.2) depends on \( \sigma_i \) and the shaded area in Fig. 1 can be either larger or smaller than \( \Delta W'' \).

5 Examples

5.1 Three-Bar Truss. Consider the truss shown in Fig. 2. The three bars of the truss have the same cross-sectional area \( A \) and they are made of the same elastic-perfectly-plastic material with Young's modulus \( E \) and yield stress \( \sigma_0 \). Let \( F \) be the applied vertical load and \( \delta \) the vertical displacement of point A. The load-displacement curve can be easily found to be (e.g., A. Phillips (1956), p. 8)

\[
F = \frac{AE\delta}{l} (1 + 2\cos^2 \alpha) \quad \text{for} \quad \frac{\sigma_0 l}{E} \leq \delta \leq \frac{\sigma_0 l}{E \cos^2 \alpha}. 
\]

(5.1.1)

\[
F = F_{\lim} = A\sigma_0 (1 + 2\cos \alpha) \quad \text{for} \quad \delta \geq \frac{\sigma_0 l}{E \cos^2 \alpha}. 
\]

(5.1.3)

After unloading, the residual stresses are

\[
\sigma_i = -2\sigma_0 \sin^2 \alpha \cos \alpha \quad \frac{1}{1 + 2\cos^3 \alpha}. 
\]

(5.1.4)

for the vertical bar, and

\[
\sigma_i = 0 \quad \frac{1}{1 + 2\cos^3 \alpha}. 
\]

(5.1.5)

for the inclined bars. Using equation (3.6) we can easily show that the stored energy is given by

\[
W'' = \frac{Al_0^2}{E} \sin^2 \alpha \cos(1 + 2\cos^3 \alpha). 
\]

(5.1.6)

The stress field of this simple structure belongs to case a) discussed in Section 3 and one can easily show by direct calculation that the stored energy is indeed equal to the shaded area in Fig. 2.

5.2 Beam Bending. Consider a rectangular beam subject to a pure bending moment \( M \). The width \( w \) of the beam is assumed to be much smaller than the thickness \( t \) so that plane stress conditions prevail. The beam is made of an elastic-perfectly-plastic material with Young's modulus \( E \) and yield stress \( \sigma_0 \).

The material is assumed to yield according to the law of von Mises with associated flow rule. The moment-curvature relation is found to be (e.g., Hodge (1959), p. 116)

\[
m = \frac{2}{3} k, \quad \text{for} \quad 0 \leq k \leq 1, 
\]

(5.2.1)

and

\[
m = 1 - \frac{1}{3k^2}, \quad \text{for} \quad k \geq 1, 
\]

(5.2.2)

where

\[
m = M/M_0 \quad \text{and} \quad k = K/K_e \quad \text{(5.2.3)}
\]

\( K \) being the curvature of the middle surface of the beam, and \( M_0 \) and \( K_e \) are given by

\[
M_0 = \frac{1}{4} \sigma_0 w t^2, \quad K_e = \frac{2\sigma_0}{E t}. 
\]

(5.2.4)

In this case the stress tensor does not rotate in stress space and belongs to case (a) discussed in Section 3. After removal of the bending moment the residual stresses can be easily calculated (e.g., see Hodge, 1959) and it is a straightforward exercise to show that the stored energy per unit length of the beam is equal to the shaded area in Fig. 3.

6 Application to Polycrystalline Aggregates

6.1 An Estimate for the Stored Energy in a Tensile Specimen. Consider a polycrystalline metal of randomly oriented single crystals subject to surface tractions that would be associated with a homogeneous stress state \( \Sigma \) and let \( E \) be the corresponding homogeneous macroscopic strain. While macroscopic quantities are homogeneous, the local stresses and strains vary from crystal to crystal as well as from point to point within the crystals. The local elastic strains are assumed to be linearly related to the local stresses in all single crystals. If \( \sigma \) and \( \epsilon \) are the local stresses and strains respectively, then

\[
V \int_0^E \epsilon : d\Sigma = W^* + \sum_c \int_{r_c} \epsilon : d\sigma : dV. 
\]

(6.1.1)

where \( W^* \) is the total elastic strain energy of the polycrystal,
that the stress state is either at a corner or at an edge of the hyperplanes defined by equation (6.1.5) in stress space. If, for a given stress state, more than one slip systems are activated, then that stress state is at an edge of the hyperplanes. The yield surface of the single crystal is the "envelope" of all single crystal yield surfaces.

Considering the next problem of simple tension of a polycrystal and let $\Sigma$ and $E$ be the macroscopic axial stress and strain, respectively. Several solutions for the uniaxial stress-strain behavior of polycrystals and composites based on the elastic-plastic properties of the single crystal constituents have been presented in the literature (Taylor, 1938a, b; Bishop and Hill, 1951; Lin, 1957; Budiansky, Hashin and Sanders, 1960; Kröner, 1961; Budiansky and Wu, 1962; Hutchinson, 1964, 1970; Berveiller and Zaoui, 1979; Chiang and Weng, 1984). A typical uniaxial stress-strain curve of a polycrystal is shown in Fig. 4. The crystals are assumed to be elastic perfectly plastic and the macroscopic hardening is a consequence of the nonuniform deformation of the grains. Budiansky and Wu (1962) analyzed the uniaxial behavior of face-centered-cubic polycrystals and their calculations indicate that only a small fraction (about 6 percent) of the grains suffers unloading during the course of their deformation, despite the fact that the applied macroscopic stress increases monotonically. Therefore, for the case of uniaxial tension equation (6.1.7) can be written as

$$\int_0^E E \, d\Sigma = \frac{W^V}{V}. \quad (6.1.8)$$

When the maximum macroscopic uniaxial strain remains small, Hutchinson’s (1964) calculations indicate that when the tensile load on the polycrystal is diminished the strain increments in all the crystals become pure elastic, i.e. unloading is linear. Therefore, using equation (6.1.8) and an argument similar to that presented in Section 3, one can show that the shaded area in Fig. 4 is approximately equal to the stored energy of the polycrystal. The effects of hardening and initial stresses in the crystals are similar to those discussed in Section 4.

After unloading, the area under the stress-strain curve shown in Fig. 4 is what one could call the macroscopic energy dissipation per unit volume of the polycrystal. However, only part of this energy is actually plastically dissipated; the rest of it is stored in the specimen increasing its internal energy. The stored energy is attributed to residual microstresses and the energy associated with the generation of new dislocations in the specimen (Bever et al. (1973)). One can argue that the stored energy is a measure of the fraction of the macroscopic energy dissipation that is not converted into heat when the material deforms plastically.

A review of the experimental as well as the theoretical developments concerning the stored energy of cold work has been given by Titchener and Bever (1958) and Bever et al. (1973). They report that the amount of stored energy depends on the extent of deformation and that strain hardening and energy storage cease at about the same stage of the deformation process. Our results agree with the experimental observations in that the shaded area in Fig. 4 increases with increasing strain until the hardening capacity of the material is exhausted.

In order to obtain an estimate for the stored energy in a tensile specimen after unloading, we consider a uniaxial stress-strain curve of the form...
The ratio $f$ is plotted as a function of $E/E_0$ in Fig. 5 for several values of the hardening exponent $N$. Our results show that the fraction of the expended energy that is stored in the specimen increases with increasing $N$ which is in agreement with experimental observations. It is interesting to note that $f \to N$ as $E \to \infty$, so that the hardening exponent $N$ provides an upper bound for the fraction of the expended energy that is stored in a tensile specimen during plastic deformation.

It should be emphasized that the aforementioned power-law relationship is such that infinite strain produces infinite stress, whereas a saturation to constant stress after large strains is more physically realistic. Therefore, in real materials, the predicted $f$ value reaches a maximum and starts decreasing when the hardening capacity of the material is exhausted. The conclusion however, that the hardening exponent $N$ provides an upper bound for the ratio of stored to expended energy is still valid.

6.2 Experimental Data and Theoretical Predictions. A complete list of the available experimental data on the stored energy of cold work is given in the review articles by Titchener and Bever (1958) and Bever et al. (1973). In most cases the ratio of stored to expended energy is less than 0.15, but some unusually large values have been reported in the literature, especially for materials with large hardening exponents. Das and Bever (1978) report that the ratio of the stored to expended energy of polycrystalline bismuth is approximately 0.80 at a strain of 0.20 and it decreases to 0.50 at a strain of 0.60. Ratios in the range 0.90 to 0.15 in cadmium (Khotkevich et al., 1954), 0.80 to 0.15 in lead (Khotkevich et al., 1954), and 0.80 to 0.27 in cadmium-lead alloys (Khotkevich and Sirenko, 1969) have been reported. The results of the present analysis provide a theoretical justification for the observed large fraction of the expended energy that is stored in the material and show that a dependence of the ratio of stored to expended energy on the amount of straining is to be expected.

Experimental results of several researchers together with the theoretical predictions for the ratio of stored to expended energy in copper tensile specimens at small strain levels are shown in Fig. 6. Electrical lead copper, purity 98.3 percent, was used in the experiments by Kunin et al. (1964); the exact compositions of the copper specimens used by Kunin (1959) and Williams (1963) were not reported. The data of Kunin (1959) and Kunin et al. (1964) were based on temperature measurements, whereas Williams’ (1963) data were obtained using a liquid-gas calorimeter. The solid line in Fig. 6 shows the prediction of the proposed model and is based on the uniaxial stress-strain curve of annealed polycrystalline copper (99.98 percent purity). The theoretical curve has a maximum at a value of the uniaxial strain about 0.10 (not shown in Fig. 6). At higher strains, the predicted values of the ratio of stored to expended energy decrease continuously, and this is due to loss of the hardening capacity of copper.

It is well established that relatively small differences in the purity of nearly pure metals have a marked effect on the stored energy (Titchener and Bever, 1958; Bever et al., 1973). This is also clear in Fig. 6, which shows that the results of three different investigations establish three distinct relations between $f$ and strain. It is also known that the strain hardening properties of copper, and, therefore, the predictions of the model, are very sensitive to small changes in the constituents and heat treatment. It should be emphasized therefore, that a direct comparison of the experimental data with the theoretical prediction shown in Fig. 6 is not possible, since each curve represents the results of a different investigation for copper with a different composition.

The predictions shown in Fig. 6 consistently overestimate the experimental data at strain levels higher than about 5 percent. The theoretical curve in Fig. 6 is based on the assumption of perfect plasticity at the single crystal level, whereas stress-strain curves of copper single crystals indicate that, at strain levels of about 5 percent or higher, substantial hardening is taking place along most of the crystal orientations (Diehl, 1956). According to the discussion in Section 4, local hardening will make the theoretical predictions only an upper bound to the actual stored energy. This provides a possible explanation for the higher $f$ values predicted by the theoretical model at high strains.
It should be also mentioned that in the proposed model the stored energy is calculated as the energy associated with residual stresses created in the metal after plastic deformation and unloading. The effects of any initial (residual) stresses left in the specimens after annealing are not taken into account. As discussed in Section 4, such stresses can play an important role, especially at small strain levels.

7 Closure
In nominally pure metals, the dominant contribution to the stored energy of cold work is the energy associated with dislocations present in excess of those in the annealed metal. The proposed theoretical model for the determination of the stored energy is based on the calculation of the energy associated with the residual stresses created in an elastic-plastic structure after inhomogeneous plastic deformation and unloading. One could argue that the stored energy calculated in such a way, is the macroscopic equivalent of the energy associated with the dislocations generated during inhomogeneous plastic deformation.

The proposed new model for the calculation of the stored energy of cold work is extremely simple and shows that the hardening exponent of the metal provides an upper bound for the ratio of stored to expended energy at large strains. For further progress towards a complete understanding of the phenomenon of the stored energy, it is necessary that detailed calculations of the energy associated with the generated dislocations be carried out. Such work is now underway.

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