Journal of Applied Mathematics and Physics (ZAMP)
Vol. 37, September 1986

0044-2275/86/005797-04 $ 2.30/0

A concise solution for shear flow problems in cylindrical geometry

By Dimitris Valougeorgis, Center for Transport Theory and Mathematical Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061, USA

I. Introduction

In two recent works [1, 2] concerning Poiseuille flow in a plane channel [1] and in a cylindrical tube [2], the F_N method [3, 4, 5] was used to provide highly accurate numerical results within the approximations of the linearized Bhatnagar-Groß-Krook (BGK) model and Maxwell's diffuse boundary conditions. The method which is easy to use can be summarized in the following way. Using the tractable full-range orthogonality properties of appropriate elementary solutions, a system of singular integral equations for the unknown distribution functions at the physical boundaries of the problem under consideration is established. The unknown quantities are then approximated by a finite expansion in terms of a set of basis functions, and the coefficients of the expansion are found by requiring the set of the reduced algebraic equations to be satisfied at certain collocation points. It is noted however that the depencence of the F_N method on the method of elementary solutions was the pitfall of any attempts to solve problems for which exact analysis is unavailable, such as internal and external binary flows in plane and cylindrical geometry.

It was realized by Garcia and Siewert [6] that the F_N method could be established in an independent and more direct way based on an integral-transform technique. Very recent work [7, 8, 9] with the F_N method was based on this more direct development and thus does not require availability of the exact elementary solutions.

It is the purpose of the present work to extend the new approach in cylindrical geometry by considering the cylindrical Poiseuille flow problem. The complete set of singular-integral equations is developed in a way that circumvents the use of the method of elementary solutions and is equivalent to the results obtained before with the conventional F_N approach [2].

II. Analysis

As discussed by Valougeorgis and Thomas [2] the cylindrical Poiseuille flow problem has been reduced to the problem of solving the integral-differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu^2}\right)\varphi(r, \mu) = -\frac{2}{\sqrt{\pi}}\int_0^\infty \varphi(r, \mu)\, e^{-\mu^2}\frac{d\mu}{\mu^2} - \frac{\sqrt{\pi}}{2}, \tag{1}$$

subject to

$$K_1(R/\mu)\,\varphi(R, \mu) + \mu K_0(R/\mu)\frac{\partial}{\partial r}\varphi(r, \mu)_{r=R} = 0. \tag{2}$$

To start our analysis we multiply Eq. (1) by $I_0(r/s)$, integrate over r from $r = 0$ to $r = R$ and manipulate the resulting equation to obtain

$$\int_0^R r I_0(r/s)\,\varphi(r,\mu)\,\mathrm{d}r + \frac{2}{\sqrt{\pi}}\frac{s^2\mu^2}{\mu^2 - s^2}t(s) = -\frac{\sqrt{\pi}}{2}\frac{s^3\mu^2}{\mu^2 - s^2}R I_1(R/s) - \frac{s^2\mu^2}{\mu^2 - s^2}R I_0(R/s)\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} + \frac{s\mu^2}{\mu^2 - s^2}R I_1(R/s)\,\varphi(R,\mu), \tag{3}$$

where

$$t(s) = \int_0^R r I_0(r/s)\int_0^\infty \varphi(r,\mu)\,e^{-\mu^2}\frac{\mathrm{d}\mu}{\mu^2}\,\mathrm{d}r.$$

Multiplying Eq. (3) by $\dfrac{e^{-\mu^2}}{\mu^2}$ and integrating over all μ we find

$$\frac{t(s)}{R}\Lambda(s) = s\int_0^\infty I_1(R/s)\,\varphi(R,\mu)\,e^{-\mu^2}\frac{\mathrm{d}\mu}{\mu^2 - s^2} - s^2\int_0^\infty I_0(R/s)\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}e^{-\mu^2}\frac{\mathrm{d}\mu}{\mu^2 - s^2} - \frac{\sqrt{\pi}}{2}s^3 I_1(R/s)\int_0^\infty\frac{e^{-\mu^2}}{\mu^2 - s^2}\,\mathrm{d}\mu, \tag{4}$$

where the dispersion function $\Lambda(s)$ is defined by

$$\Lambda(s) = 1 + \frac{2}{\sqrt{\pi}}s^2\int_0^\infty\frac{e^{-\mu^2}}{\mu^2 - s^2}\,\mathrm{d}\mu.$$

It is easy to show that $\Lambda(s)$ has no zeros in the finite cut plane. However, since $\Lambda(s) = 0$ for $|s|$ tending to infinity we may deduce one "discrete" singular integral equation. In the limit $|s| \to \infty$, we obtain from Eq. (4) after we use Eq. (2) that

$$\int_0^\infty\frac{K_1(R/\mu)}{K_0(R/\mu)}\varphi(R,\mu)\,e^{-\mu^2}\frac{\mathrm{d}\mu}{\mu} = \frac{\pi}{8}R. \tag{5}$$

We now consider the continuous spectrum $s = \xi$, $\xi \in (0,\infty)$. The left and right hand sides of Eq. (4) are analytic in the complex cut plane from $-\infty$ to ∞ along the real axis. We let s approach the branch cut and using Plemelj formulas [10] we find

$$\begin{aligned}\frac{t(s)}{R}&\{p(\xi) \pm i2\sqrt{\pi}\,\xi^2 e^{-\xi^2}\}\\ &= \xi\int_0^\infty I_1(R/\xi)\,\varphi(R,\mu)\,Pv\left(\frac{1}{\mu^2 - \xi^2}\right)e^{-\mu^2}\mathrm{d}\mu \pm i\pi\xi I_1(R/\xi)\,\varphi(R,\xi)\,e^{-\xi^2}\\ &\quad - \xi^2\int_0^\infty I_0(R/\xi)\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}Pv\left(\frac{1}{\mu^2 - \xi^2}\right)e^{-\mu^2}\mathrm{d}\mu \mp i\pi\xi^2 I_0(R/\xi)\left.\frac{\partial\varphi}{\partial r}\right|_{r=R}e^{-\xi^2}\\ &\quad - \frac{\sqrt{\pi}}{2}\xi^3 I_1(R/\xi)\int_0^\infty Pv\left(\frac{1}{\mu^2 - \xi^2}\right)e^{-\mu^2}\mathrm{d}\mu \mp i\frac{\pi^{3/2}}{2}\xi^3 I_1(R/\xi)\,e^{-\xi^2},\end{aligned} \tag{6}$$

where

$$p(\xi) = 1 + \frac{2}{\pi}\xi^2\int_0^\infty Pv\left(\frac{1}{\mu^2 - \xi^2}\right)e^{-\mu^2}\mathrm{d}\mu.$$

In Eq. (6) the Pv indicates that integrals are to be interpreted in the Cauchy principal-value sense. Eliminating $t(s)$ from Eq. (6) and substituting Eq. (2) into the resulting equation yields

$$\begin{aligned}
&\frac{1}{\xi}\left\{I_1(R/\xi) + I_0(R/\xi)\frac{K_1(R/\xi)}{K_0(R/\xi)}\right\}\varphi(R,\xi)\,e^{-\xi^2}\lambda(\xi)\\
&= \xi\int_0^\infty I_1(R/\xi)\,\varphi(R,\mu)\,e^{-\mu^2}\left[Pv\left(\frac{1}{\mu-\xi}\right)+\frac{1}{\mu+\xi}\right]\frac{d\mu}{\mu}\\
&+ \xi^2\int_0^\infty I_0(R/\xi)\frac{K_1(R/\mu)}{K_0(R/\mu)}\varphi(R,\mu)\,e^{-\mu^2}\left[Pv\left(\frac{1}{\mu-\xi}\right)+\frac{1}{\mu+\xi}\right]\frac{d\mu}{\mu^2}\\
&+ \frac{\sqrt{\pi}}{2}\xi I_1(R/\xi)\,\lambda(\xi)\,e^{-\xi^2} - \frac{\sqrt{\pi}}{2}\xi^2 I_1(R/\xi)\int_0^\infty\left[Pv\left(\frac{1}{\mu-\xi}\right)-\frac{1}{\mu+\xi}\right]d\mu, \qquad (7)
\end{aligned}$$

where

$$\lambda(\xi) = \sqrt{\pi}\,p(\xi)\,e^{\xi^2}.$$

We divide Eq. (7) by $I_1(R/\xi)$ to deduce

$$\int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)]\left[\mu + \xi\frac{I_0(R/\xi)\,K_1(r/\mu)}{I_1(R/\xi)\,K_0(R/\mu)}\right]\varphi(R/\mu)\,e^{-\mu^2}\frac{d\mu}{\mu^2} = \frac{\pi}{2}\xi, \qquad (8)$$

with

$$f(\xi,\mu) = Pv\left(\frac{\xi}{\xi-\mu}\right) + \lambda(\xi)\,\delta(\xi-\mu).$$

Equations (8) and (5) constitute a complete set of singular integral equations upon which the F_N solution is based. These equations are identical with Eqs. (20) and (21) obtained by Valougeorgis and Thomas [2], using full-range orthogonality properties of the eigenfunctions $f(\xi,\mu)$, $\xi\in(-\infty,\infty)$ and $f_\infty = 1$.

III. Conclusions

As noted in the foregoing discussion, using an integral-transform technique, we now have developed a complete formalism for the unknown function $\varphi(r,\mu)$ at $r = R$, without invoking any aspects of the elementary solutions of Eq. (1). It is this advantage alone which makes this fresh approach of the F_N method promising for applications to more difficult problems not amenable to exact analysis. We are certain that the present analysis can be extended in the fields of neutron transport theory and radiative transfer.

Acknowledgement

The author is grateful to Professor J. R. Thomas Jr. for several helpful suggestions. This work was supported in part by Department of Energy Grant No. DE-AS05-80ER10711 and National Science Foundation Grant No. DMS-8312451.

References

[1] C. E. Siewert, R. D. Garcia and P. Grandjean, J. Math. Phys. *21*, 2760 (1980).
[2] D. Valougeorgis and J. R. Thomas, Jr., Phys. Fluids *29*, 423 (1986).

[3] C. E. Siewert and P. Benoist, Nucl. Sci. Eng. *69*, 154 (1979).
[4] P. Grandjean and C. E. Siewert, Nucl. Sci. Eng. *69*, 161 (1979).
[5] S. K. Loyalka and J. H. Ferziger, Phys. Fluids *10*, 1833 (1967).
[6] R. D. M. Garcia and C. E. Siewert, J. Comp. Phys. *46*, 237 (1982).
[7] C. E. Siewert, Z. angew. Math. Phys. *35*, 144 (1984).
[8] C. E. Siewert and J. R. Thomas, Jr., in press (1986).
[9] D. Valougeorgis, will be presented at the 15th International Symposium on Rarefied Gas Dynamics, 1986.
[10] N. I. Muskhelishvili, Singular Integral Equations, Noordhoff, Groningen, The Netherlands 1953.

Abstract

A new development on the F_N method for solving cylindrical flows problems in the field of rarefied gas dynamics is presented. The present approach is based on an integral-transform technique of the linearized Bhatnagar-Gross-Krook (BGK) model for shear flow and thus does not require availability of the exact elementary solutions of the shear flow equation.

Zusammenfassung

Ein neues Vorgehen bei der Anwendung der F_N-Methode zur Berechnung der Strömung verdünnter Gase in zylindrischer Geometrie wird vorgestellt. Eine Integral-Transformations-Technik wird auf das linearisierte BGK Modell der Scherströmung angewendet. Dadurch benötigt das Verfahren die exakten Elementarlösungen der Strömungsgleichungen nicht.

(Received: February 5, 1986)