Exact numerical results for Poiseuille and thermal creep flow in a cylindrical tube

Dimitris Valougeorgis and J. R. Thomas, Jr.
Mechanical Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061

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The F_N method is used, in the field of rarefied gas dynamics, to develop a complete solution for the cylindrical Poiseuille flow and thermal creep problems. The linearized Bhatnagar–Gross–Krook (BGK) model and purely diffuse reflection at the surface are used to describe the physical problem. The derived set of singular integral equations is solved by polynomial expansion and collocation. By choosing suitable F_N approximations, the solution of both problems under consideration is accomplished with a single matrix inversion, minimizing computational time and effort. The converged numerical results for the flow rates and the velocity profiles are correct to four significant figures, thus supporting the results of previous authors achieved by other methods.

I. INTRODUCTION

Exact analysis of slip-flow problems in the kinetic theory of gases, based on the method of elementary solutions, was presented first by Cercignani.[1,2] He adapted the method of elementary solutions,[1] that originated in neutron transport theory, to solve a number of interesting problems in rarefied gas dynamics, including the problem of plane Poiseuille flow.[2] Cercignani later considered cylindrical Poiseuille flow and presented results based on a direct numerical approach to the integral form of the Bhatnager–Gross–Krook (BGK) model[3] and on a variational technique.[4] The method of elementary solutions was used by Ferziger[5] to derive analytical results for the cylindrical Poiseuille problem in the near-free-molecule and near-continuum regimes. This work was extended by Loyalka[6] to the thermal creep problem in a cylindrical tube, indicating that earlier results of Sone and Yamamoto[7] were in error. In 1975, Loyalka[8] presented a first complete solution of the thermal transpiration problem in plane and cylindrical geometry within the approximations of the BGK model and Maxwellian boundary conditions. These results, obtained through numerical solution of the integral form of the particle transport equation, were in good agreement with earlier work by Cercignani,[4] while more recently reported variational results[9–11] for the Poiseuille flow problem appeared to be inaccurate (off by 10%–15%). Loyalka's work was extended in several recent papers[12–14] for both plane and cylindrical geometry and a variety of collisional models. Numerical results were obtained and compared with experimental data. However, numerical results that can be considered numerically "exact" were provided in plane geometry only.[15] Loyalka, Petrellis, and Storvick[15] used the method of elementary solutions to derive Fredholm integral equations for Couette, Poiseuille, and thermal creep flow problems with the Maxwell diffuse-specular boundary conditions. By iterating these equations, they provided highly accurate numerical results in plane geometry. No similarly accurate results were reported for the cylindrical case.

Quite recently, Siewert, Garcia, and Grandjean[16] used the F_N method[17–19] to compute highly accurate flow rates for Poiseuille flow in a plane channel. These results agreed with those of Loyalka, Petrellis, and Storvick[15] to five significant figures even though they were computed with a relatively low-order approximation. The F_N method has been extended to cylindrical geometry for neutron transport problems by Thomas, Southers, and Siewert[20] and Siewert and Thomas.[21,22]

In this paper we report application of the F_N method to the problems of Poiseuille flow and thermal creep in a cylindrical tube and provide highly accurate results within the approximation of the linearized BGK model and Maxwell's diffuse boundary conditions. Our converged numerical results, which we believe to be accurate to within ± 1 in the last significant figure shown, are in agreement with those of Loyalka[8] to within two to four significant figures. The principal value of these results is as a test of the accuracy of solution methods used previously for the BGK model, some of which have recently been extended to higher-order models and models for polyatomic gases[12–14].

II. BASIC ANALYSIS

Following Ferziger[5] and Loyalka,[6] we consider the integral equation

$$Z(r)=\frac{2}{\sqrt{\pi}}\int_0^\infty d\mu\,\frac{e^{-\mu^2}}{\mu^2}\left[\int_0^r tZ(t)K_0\left(\frac{r}{\mu}\right)I_0\left(\frac{t}{\mu}\right)dt+\int_r^R tZ(t)K_0\left(\frac{t}{\mu}\right)I_0\left(\frac{r}{\mu}\right)dt\right]+Y_{P,PT}(r),\quad(1)$$

where I_0 and K_0 represent modified Bessel functions of the first and second kind, respectively. The function $Z(r)$ is related to the velocity profiles in the Poiseuille (P) and thermal creep (T) problems through[6]

$$Z_P(r)=\sqrt{\pi}\,[v_P(r)+\tfrac{1}{2}]\quad(2)$$

and

$$Z_{PT}(r) = \sqrt{\pi}\,[v_{PT}(r) + \tfrac{1}{4}], \tag{3}$$

where

$$v_{PT}(r) = v_p(r) - v_T(r). \tag{4}$$

In Eq. (1),

$$Y_P(r) = \sqrt{\pi}/2, \tag{5}$$

while

$$Y_{PT}(r) = \frac{\sqrt{\pi}}{4} + \int_0^\infty d\mu\, e^{-\mu^2}\left[\int_0^r tK_0\left(\frac{r}{\mu}\right) I_0\left(\frac{t}{\mu}\right) dt + \int_r^R tK_0\left(\frac{t}{\mu}\right) I_0\left(\frac{r}{\mu}\right) dt\right]. \tag{6}$$

Performing the integration over t in Eq. (6) and using a standard identity for the Bessel functions,[23] we find

$$Y_{PT}(r) = \frac{\sqrt{\pi}}{2} - R\int_0^\infty \mu I_0\left(\frac{r}{\mu}\right) K_1\left(\frac{R}{\mu}\right) e^{-\mu^2} d\mu. \tag{7}$$

If we define[20]

$$\phi(r,\mu) = K_0\left(\frac{r}{\mu}\right)\int_0^r tZ(t) I_0\left(\frac{t}{\mu}\right) dt + I_0\left(\frac{r}{\mu}\right)\int_r^R tZ(t) K_0\left(\frac{t}{\mu}\right) dt, \tag{8}$$

and differentiate twice, we find that $\phi(r,\mu)$ must satisfy the integrodifferential equations

$$(B\phi_p)(r,\mu) = -\sqrt{\pi}/2, \tag{9}$$

and

$$(B\phi_{PT})(r,\mu) = -\frac{\sqrt{\pi}}{2} + R\int_0^\infty \mu I_0\left(\frac{r}{\mu}\right) K_1\left(\frac{R}{\mu}\right) e^{-\mu^2} d\mu. \tag{10}$$

Here the subscripts P and PT have the same meanings as in Eqs. (2) and (3), and the operator B is defined as

$$(Bf)(r,\mu) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu^2}\right) f(r,\mu) + \frac{2}{\sqrt{\pi}}\int_0^\infty f(r,\mu) e^{-\mu^2}\frac{d\mu}{\mu^2}. \tag{11}$$

It is also a consequence of Eq. (8) that both $\phi_P(r,\mu)$ and $\phi_{PT}(r,\mu)$ must satisfy the boundary condition

$$K_1\left(\frac{R}{\mu}\right)\phi(R,\mu) + \mu K_0\left(\frac{R}{\mu}\right)\frac{\partial\phi}{\partial r}(r,\mu)\Big|_{r=R} = 0. \tag{12}$$

At this point we find it convenient to introduce

$$Y(r,\mu) = \phi_{PT}(r,\mu) - (\sqrt{\pi}/2)R\,\mu^3 K_1(R/\mu) I_0(r/\mu), \tag{13}$$

so that $Y(r,\mu)$ is the solution of

$$(BY)(r,\mu) = -\sqrt{\pi}/2, \tag{14}$$

which is analogous to Eq. (9), subject to the modified boundary condition

$$K_1\left(\frac{R}{\mu}\right) Y(R,\mu) + \mu K_0\left(\frac{R}{\mu}\right)\frac{\partial Y}{\partial r}(r,\mu)\Big|_{r=R} = -\frac{\sqrt{\pi}}{2}\mu^4 K_1\left(\frac{R}{\mu}\right). \tag{15}$$

Thus, both Poiseuille (P) and pseudothermal creep (PT) problems have been reduced to the problem of solving the same integrodifferential equation subject to different boundary conditions. In addition, both $\phi_P(r,\mu)$ and $Y(r,\mu)$ clearly must remain bounded for $r \to 0$ for all values of μ. The general solution of Eq. (9) or Eq. (14) that satisfies this condition can be expressed in terms of the elementary solutions as

$$X(r,\mu) = \mu^2\left[C_0 f_\infty + \int_0^\infty C(\xi)[f(\xi,\mu) + f(-\xi,\mu)] \times I_0\left(\frac{r}{\xi}\right) d\xi\right] + g(r,\mu), \tag{16}$$

where $X(r,\mu)$ represents $\phi_p(r,\mu)$ or $Y(r,\mu)$, C_0 and $C(\xi)$ are expansion coefficients to be determined, and

$$f(\xi,\mu) = \mathrm{Pv}(\xi/\xi - \mu) + \lambda(\xi)\delta(\xi - \mu), \tag{17}$$

with[3]

$$\lambda(\xi) = e^{\xi^2}\left[\sqrt{\pi} + \int_{-\infty}^\infty \mathrm{Pv}\left(\frac{\xi}{t-\xi}\right) e^{-t^2} dt\right]. \tag{18}$$

In Eq. (17), $\delta(x)$ represents the Dirac delta functional and the Pv indicates that integrals are to be interpreted in the Cauchy principal-value sense. The particular solution

$$g(r,\mu) = -(\sqrt{\pi}/4)\mu^2(r^2 - R^2 + 4\mu^2) \tag{19}$$

may be verified by direct substitution into Eqs. (9) or (14).

III. THE F_N SOLUTION

Cercignani[1] proved orthogonality properties for the eigenfunctions $f(\xi,\mu)$, $\xi \in (-\infty,\infty)$, and $f_\infty = 1$. We use these, in the manner explained by Thomas, Southers, and Siewert[20] to derive the singular integral equations

$$\int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)]\left(\mu + \xi\frac{I_0(R/\xi)}{I_1(R/\xi)}\frac{K_1(R/\mu)}{K_0(R/\mu)}\right) \times \phi_P(R,\mu)\frac{e^{-\mu^2} d\mu}{\mu^2} = \frac{\pi\xi}{2} \tag{20}$$

and

$$\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)}\phi_P(R,\mu)\frac{e^{-\mu^2} d\mu}{\mu} = \frac{\pi}{8}R, \tag{21}$$

for the Poiseuille problem. Similarly, the function $Y(r,\mu)$ is seen to satisfy the singular integral equations

$$\int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)]\left(\mu + \xi\frac{I_0(R/\xi)}{I_1(R/\xi)}\frac{K_1(R/\mu)}{K_0(R/\mu)}\right) \times Y(R,\mu)\frac{e^{-\mu^2} d\mu}{\mu^2} = \frac{\pi\xi}{2} - \frac{\sqrt{\pi}}{2}\xi\frac{I_0(R/\xi)}{I_1(R/\xi)} \times \int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)]\frac{K_1(R/\mu)}{K_0(R/\mu)}\mu^2 e^{-\mu^2} d\mu \tag{22}$$

and

$$\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)} Y(R,\mu)\frac{e^{-\mu^2}}{\mu} d\mu = \frac{\pi R}{8} - \frac{\sqrt{\pi}}{2}\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)}\mu^3 e^{-\mu^2} d\mu. \tag{23}$$

We substitute the two F_N approximations,

$$\phi_P(R,\mu) = \mu K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\sum_{\alpha=0}^N A_\alpha \mu^\alpha \tag{24}$$

and

$$Y(R,\mu) = -\frac{\sqrt{\pi}}{2}\mu^4 + \mu K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\sum_{\alpha=0}^{N} B_\alpha \mu^\alpha, \tag{25}$$

into Eqs. (20)–(23) to find

$$\sum_{\alpha=0}^{N}\left(E_\alpha(\xi) + \xi\frac{I_0(R/\xi)}{I_1(R/\xi)}D_\alpha(\xi)\right)A_\alpha = \frac{\pi\xi}{2}, \tag{26}$$

$$\sum_{\alpha=0}^{N} S_\alpha A_\alpha = \frac{\pi R}{8}, \tag{27}$$

$$\sum_{\alpha=0}^{N}\left(E_\alpha(\xi) + \xi\frac{I_0(R/\xi)}{I_1(R/\xi)}D_\alpha(\xi)\right)B_\alpha = \frac{\pi\xi}{4}, \tag{28}$$

and

$$\sum_{\alpha=0}^{N} S_\alpha B_\alpha = \frac{\pi R}{8}. \tag{29}$$

The functions D_α, E_α, and S_α are defined as

$$D_\alpha(\xi) = \int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)] \times K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\mu^{\alpha-1}e^{-\mu^2}\,d\mu, \tag{30}$$

$$E_\alpha(\xi) = \int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)] \times K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\mu^{\alpha}e^{-\mu^2}\,d\mu, \tag{31}$$

and

$$S_\alpha = \int_0^\infty K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\mu^{\alpha}e^{-\mu^2}\,d\mu. \tag{32}$$

Substituting Eq. (17) for $f(\xi,\mu)$, we find that $D_\alpha(\xi)$ and $E_\alpha(\xi)$ may be computed for $\xi\epsilon[0,\infty)$, from the explicit formulas

$$\begin{aligned} D_\alpha(\xi) = {} & \xi^{\alpha-1}K_1\left(\frac{R}{\xi}\right) I_1\left(\frac{R}{\xi}\right)\left(\sqrt{\pi} - \xi\int_0^\infty \frac{e^{-\mu^2}d\mu}{\mu+\xi}\right) \\ & - \xi^\alpha \int_0^\infty \frac{K_1(R/\xi)I_1(R/\xi) - K_1(R/\mu)I_1(R/\mu)}{\xi-\mu} \\ & \times e^{-\mu^2}d\mu + (-\xi)^\alpha \int_0^\infty K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right) \\ & \times \frac{e^{-\mu^2}}{\mu+\xi}d\mu - \int_0^\infty K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right) \\ & \times \sum_{m=0}^{\alpha-2}\mu^m\xi^{\alpha-1-m}[1+(-1)^{\alpha-2-m}]e^{-\mu^2}\,d\mu, \end{aligned} \tag{33}$$

and

$$\begin{aligned} E_\alpha(\xi) \\ = {} & \xi^{\alpha}K_0\left(\frac{R}{\xi}\right) I_1\left(\frac{R}{\xi}\right)\left(\sqrt{\pi} - \xi\int_0^\infty \frac{e^{-\mu^2}}{\xi+\mu}d\mu\right) - \xi^{\alpha+1} \\ & \times \int_0^\infty \frac{K_0(R/\xi)I_1(R/\xi) - K_0(R/\mu)I_1(R/\mu)}{\xi-\mu} \\ & \times e^{-\mu^2}d\mu + (-\xi)^{\alpha+1} \int_0^\infty K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right) \\ & \times \frac{e^{-\mu^2}}{\mu+\xi}d\mu - \int_0^\infty K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right) \\ & \times \sum_{m=0}^{\alpha-1}\mu^m\xi^{\alpha-m}[1+(-1)^{\alpha-1-m}]e^{-\mu^2}\,d\mu, \end{aligned} \tag{34}$$

or can be readily generated from the recursion relations

$$D_\alpha(\xi) = \xi^2 D_{\alpha-2}(\xi) - 2\xi\int_0^\infty K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\mu^{\alpha-2}e^{-\mu^2}\,d\mu, \tag{35}$$

and

$$E_\alpha(\xi) = \xi^2 E_{\alpha-2}(\xi) - 2\xi\int_0^\infty K_0\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right)\mu^{\alpha-1}e^{-\mu^2}\,d\mu. \tag{36}$$

Starting values may be obtained from Eqs. (33) and (34). We now have $N+1$ unknowns A_α, $\alpha = 0,1,\ldots,N$ for the Poiseuille flow problem, and B_α, $\alpha = 0,1,\ldots,N$ for the thermal creep problem. Evaluating Eqs. (26) and (28) at N distinct values of $\xi\epsilon[0,\infty)$ leads to a system of $N+1$ linear algebraic equations in each case:

$$\sum_{\alpha=0}^{N}\left(E_\alpha(\xi_\beta) + \xi_\beta\frac{I_0(R/\xi_\beta)}{I_1(R/\xi_\beta)}D_\alpha(\xi_\beta)\right)A_\alpha = \frac{\pi\xi}{2}, \quad \beta = 1,2,\ldots,N, \tag{37}$$

and

$$\sum_{\alpha=0}^{N} S_\alpha A_\alpha = \frac{\pi R}{8}, \tag{38}$$

for the Poiseuille problem, and

$$\sum_{\alpha=0}^{N}\left(E_\alpha(\xi_\beta) + \xi_\beta\frac{I_0(R/\xi_\beta)}{I_1(R/\xi_\beta)}D_\alpha(\xi_\beta)\right)B_\alpha = \frac{\pi\xi}{4}, \quad \beta = 1,2,\ldots,N, \tag{39}$$

and

$$\sum_{\alpha=0}^{N} S_\alpha B_\alpha = \frac{\pi R}{8}, \tag{40}$$

for the "*PT*" problem. It is important to note that because of the forms (24) and (25) for the F_N approximation, the systems (37) and (38) and (39) and (40) are identical except for the right-hand sides. Thus, the solution of both sets is accomplished with a single matrix inversion. These equations may then be solved straightforwardly to find the required constants A_α, B_α, $\alpha = 0,1,\ldots,N$ and consequently yield explicit results for $\phi_P(R,\mu)$ and $Y(R,\mu)$ through Eqs. (24) and (25). These results may, in turn, be used to obtain flow rates and velocity profiles in terms of the expansion coefficients C_0 and $C(\xi)$ or surface quantities only.

IV. MACROSCOPIC QUANTITIES

Using Eqs. (1), (2), and (8), the velocity profile for the Poiseuille problem may be expressed as

$$v_P(r) = \frac{2}{\pi}\int_0^\infty \phi_P(r,\mu)\frac{e^{-\mu^2}}{\mu^2}d\mu \tag{41}$$

and the volumetric flow rate as

$$Q_P = \frac{4}{R^3}\int_0^R v_P(r)r\,dr. \tag{42}$$

If we define

$$\phi_n(r) = \int_0^\infty \phi(r,\mu)\,\mu^n e^{-\mu^2}\frac{d\mu}{\mu^2} \quad (43)$$

and take moments of Eq. (9), we find

$$Q_P = \left(\frac{R}{4} - \frac{1}{R}\right) + \frac{8}{\pi}\left(\frac{1}{R}\phi_2(R) - \frac{2}{R^2}\frac{d\phi_4}{dr}\bigg|_{r=R}\right), \quad (44)$$

which may be written as

$$Q_P = \frac{R}{4} - \frac{1}{R} + \frac{8}{\pi}\left(\frac{1}{R}\int_0^\infty \phi(R,\mu)e^{-\mu^2}d\mu + \frac{2}{R^2}\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)}\phi(R,\mu)\mu e^{-\mu^2}d\mu\right), \quad (45)$$

after application of the boundary condition (12).

By a similar procedure we find

$$Q_{PT} = \frac{R}{4} - \frac{1}{2R} + \frac{8}{\pi}\left(\frac{1}{R}\int_0^\infty Y(R,\mu)e^{-\mu^2}d\mu + \frac{2}{R^2}\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)}Y(R,\mu)e^{-\mu^2}d\mu + \frac{\sqrt{\pi}}{R^2}\int_0^\infty \frac{K_1(R/\mu)}{K_0(R/\mu)}\mu^5 e^{-\mu^2}d\mu\right). \quad (46)$$

Finally, we substitute the F_N approximations given by Eqs. (24) and (25) into Eqs. (45) and (46) to express the flow rates simply in terms of surface quantities as

$$Q_P = \frac{R}{4} - \frac{1}{R} + \frac{8}{\pi}\sum_{\alpha=0}^{N} A_\alpha \times\left[\frac{1}{R}\int_0^\infty K_0\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\mu^{\alpha+1}e^{-\mu^2}d\mu + \frac{2}{R^2}\int_0^\infty K_1\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\mu^{\alpha+2}e^{-\mu^2}d\mu\right] \quad (47)$$

and

$$Q_{PT} = \frac{R}{4} - \frac{2}{R} + \frac{8}{\pi}\sum_{\alpha=0}^{N} B_\alpha \times\left[\frac{1}{R}\int_0^\infty K_0\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\mu^{\alpha+1}e^{-\mu^2}d\mu + \frac{2}{R^2}\int_0^\infty K_1\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\mu^{\alpha+2}e^{-\mu^2}d\mu\right], \quad (48)$$

with

$$Q_T = Q_P - Q_{PT}. \quad (49)$$

We can also show straightforwardly that the velocities at the surface are given by

$$v_P(R) = \frac{2}{\pi}\sum_{\alpha=0}^{N} A_\alpha\int_0^\infty K_0\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\times\mu^{\alpha-1}e^{-\mu^2}d\mu, \quad (50)$$

and

$$v_{PT}(R) = \frac{2}{\pi}\sum_{\alpha=0}^{N} B_\alpha\int_0^\infty K_0\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\times\mu^{\alpha-1}e^{-\mu^2}d\mu, \quad (51)$$

where

$$v_T(R) = v_P(R) - v_{PT}(R). \quad (52)$$

For a complete solution of the problem we need the functions $\phi_p(r,\mu)$ and $Y(r,\mu)$ for all r, and thus we proceed by establishing the expansion coefficients $C(\xi)$ and C_0. Full-range orthogonality[1] may be used to obtain

$$C(\xi) = [I_0(R/\xi)N(\xi)]^{-1}\times\left(\int_0^\infty [f(\xi,\mu) - f(\xi,-\mu)]\times\phi_P(R,\mu)\frac{e^{-\mu^2}}{\mu}d\mu - \frac{\pi\xi}{2}\right), \quad (53)$$

where

$$N(\xi) = \xi e^{\xi^2}[\pi\lambda^2(\xi) + \pi^2\xi^2 e^{-2\xi^2}], \quad (54)$$

and

$$C_0 = \frac{3\sqrt{\pi}}{2} + \frac{4}{\sqrt{\pi}}\int_0^\infty \phi_P(R,\mu)e^{-\mu^2}d\mu. \quad (55)$$

Substituting the F_N approximation (24) into Eqs. (53) and (55) yields

$$C(\xi) = \left[I_0\left(\frac{R}{\xi}\right)N(\xi)\right]^{-1}\left(\sum_{\alpha=0}^{N} A_\alpha E_\alpha(\xi) - \frac{\pi\xi}{2}\right) \quad (56)$$

and

$$C_0 = \frac{3\sqrt{\pi}}{2} + \frac{4}{\sqrt{\pi}}\sum_{\alpha=0}^{N} A_\alpha R_\alpha, \quad (57)$$

where

$$R_\alpha = \int_0^\infty K_0\left(\frac{R}{\mu}\right)I_1\left(\frac{R}{\mu}\right)\mu^{\alpha+1}e^{-\mu^2}d\mu. \quad (58)$$

The velocity profile $v_P(r)$ may be expressed in terms of the F_N coefficients. By substituting the general solution given by Eq. (16) into Eq. (41) we obtain

$$v_P(r) = \tfrac{1}{4}(R^2 - r^2) + \frac{C_0}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\int_0^\infty C(\xi)I_0\left(\frac{r}{\xi}\right)d\xi - \frac{1}{2}. \quad (59)$$

We similarly obtain $v_T(r)$ through

$$v_{PT}(r) = \frac{1}{4}(R^2 - r^2) + \frac{C_0}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}\int_0^\infty C(\xi)I_0\left(\frac{r}{\xi}\right)d\xi - \frac{1}{4}, \quad (60)$$

and Eq. (4).

Clearly, alternative expressions for Q_P, Q_T, $v_P(R)$, and $v_T(R)$ can be derived in this same manner and used as numerical checks for the previous expressions.

V. NUMERICAL RESULTS

For a given value of N we first choose a set of collocation points $\{\xi_\beta\}$ such that $0 < \xi_1 < \cdots < \xi_N < \infty$. We have found the positive zeros of the Hermite polynomials of degree $2N$ to be an effective choice. We then compute the functions $D_\alpha(\xi_\beta)$ and $E_\alpha(\xi_\beta)$ from Eqs. (33) and (34) or Eqs. (35) and (36) and the function S_α from Eq. (32). To find the desired values of some of these integrals requires the use of l'Hospital's rule when $\xi = \mu$. Ferziger[5] has suggested a

TABLE I. Convergence of the Poiseuille flow rate $Q_P(R)$.

R	Order of the approximation 4	8	12	16	20	22	24	26	28	30	Converged results	Ref. 8
0.0001	1.5038	1.5038									1.5038	...
0.001	1.4995	1.4995									1.4995	1.5013
0.01	1.4760	1.4760									1.4760	1.4763
0.02	1.4597	1.4597	1.4598	1.4598							1.4598	1.4601
0.03	1.4474	1.4475	1.4475	1.4475	1.4475	1.4475	1.4475	1.4475	1.4476	1.4476	1.4476	1.4481
0.04	1.4375	1.4375	1.4376	1.4376	1.4377	1.4377					1.4377	1.4384
0.05	1.4292	1.4293	1.4293	1.4294	1.4294	1.4295	1.4295				1.4295	1.4303
0.07	1.4159	1.4161	1.4162	1.4163	1.4164	1.4164	1.4165	1.4165			1.4165	1.4175
0.09	1.4057	1.4060	1.4062	1.4063	1.4064	1.4065	1.4065	1.4066	1.4066		1.4066	1.4077
0.1	1.4015	1.4018	1.4020	1.4022	1.4024	1.4024	1.4025	1.4025	1.4026	1.4026	1.4026	1.4037
0.3	1.3715	1.3730	1.3739	1.3744	1.3748	1.3750	1.3751	1.3752	1.3753	1.3754	1.3754	1.3759
0.5	1.3819	1.3842	1.3852	1.3858	1.3862	1.3863	1.3864	1.3864			1.3864	1.3863
0.7	1.4066	1.4090	1.4099	1.4103	1.4104	1.4105	1.4105				1.4105	1.4101
0.9	1.4384	1.4405	1.4411	1.4413	1.4413						1.4413	1.4408
1.0	1.4559	1.4577	1.4582	1.4583	1.4583						1.4583	1.4578
1.25	1.5027	1.5039	1.5041	1.5041							1.5041	1.5035
1.5	1.5524	1.5531	1.5532	1.5532							1.5532	1.5526
2.0	1.6574	1.6576	1.6576								1.6576	1.6571
2.5	1.7670	1.7671	1.7671	1.7672	1.7672	1.7672					1.7672	1.7667
3.0	1.8798	1.8799	1.8799	1.8800	1.8800						1.8800	1.8796
3.5	1.9948	1.9949	1.9949	1.9950	1.9950						1.9950	1.9948
4.0	2.1114	2.1115	2.1116	2.1116							2.1116	2.1117
5.0	2.3481	2.3482	2.3483	2.3483							2.3483	2.3493
6.0	2.5880	2.5881	2.5882	2.5882							2.5882	2.5906
7.0	2.83005	2.83017	2.83021	2.83022	2.83024	2.83024					2.8302	2.8346
9.0	3.31835	3.31847	3.31850	3.31851	3.31853	3.31853					3.3185	3.3291
10.0	3.56394	3.56405	3.56408	3.56409	3.56411	3.56411					3.5641	3.5791
100.0	26.0215	26.0216	26.0216								26.021	...

TABLE II. Convergence of the thermal creep flow rate $Q_T(R)$.

R	Order of the approximation 4	8	12	16	20	22	24	26	28	Converged results	Ref. 8
0.0001	0.7515	0.7515								0.7515	...
0.001	0.7466	0.7466								0.7466	0.7466
0.01	0.7177	0.7177								0.7177	0.7177
0.02	0.6956	0.6956								0.6956	0.6958
0.03	0.6777	0.6778	0.6778							0.6778	0.6780
0.04	0.6624	0.6624	0.6624	0.6625	0.6625					0.6625	0.6628
0.05	0.6488	0.6489	0.6489	0.6489	0.6489	0.6489	0.6490	0.6490		0.6490	0.6493
0.07	0.6253	0.6254	0.6255	0.6255	0.6256	0.6256				0.6256	0.6261
0.09	0.6054	0.6055	0.6056	0.6056	0.6057	0.6057	0.6058	0.6058		0.6058	0.6063
0.1	0.5964	0.5965	0.5966	0.5967	0.5967	0.5968	0.5968			0.5968	0.5973
0.3	0.4806	0.4812	0.4815	0.4817	0.4819	0.4819	0.4820	0.4820	0.4821	0.4821	0.4823
0.5	0.4155	0.4162	0.4166	0.4168	0.4169	0.4169	0.4170	0.4170		0.4170	0.4169
0.7	0.3702	0.3709	0.3711	0.3712	0.3713	0.3713	0.3713			0.3713	0.3712
0.9	0.3357	0.3362	0.3363	0.3364	0.3364					0.3364	0.3363
1.0	0.3211	0.3216	0.3217	0.3217						0.3217	0.3216
1.25	0.2903	0.2906	0.2906	0.2906	0.2906	0.2906	0.2907	0.2907		0.2907	0.2906
1.5	0.2654	0.2655	0.2656	0.2656						0.2656	0.2655
2.0	0.2270	0.2271	0.2271							0.2271	0.2271
2.5	0.1986	0.1986	0.1986	0.1987	0.1987					0.1987	0.1987
3.0	0.1766	0.1766								0.1766	0.1767
3.5	0.1589	0.1590	0.1590							0.1590	0.1591
4.0	0.1445	0.1445								0.1445	0.1447
5.0	0.1222	0.1222								0.1222	0.1224
6.0	0.1058	0.1058								0.1058	0.1060
7.0	0.09319	0.09321	0.09322	0.09322						0.09322	0.0934
9.0	0.07521	0.07522	0.07522	0.07522	0.07523	0.07523				0.07523	0.0755
10.0	0.06856	0.06857	0.06858	0.06858						0.06858	0.0688
100.0	0.007581	0.007582	0.007583	0.007583						0.007583	...

TABLE III. Velocity slip at the wall.

R	Poiseuille flow $v_P(R)$	Thermal creep flow $v_T(R)$
0.0001	0.5639(− 4)	0.2817(− 4)
0.001	0.5615(− 3)	0.2792(− 3)
0.01	0.5484(− 2)	0.2647(− 2)
0.03	0.1594(− 1)	0.7340(− 2)
0.05	0.2597(− 1)	0.1151(− 1)
0.07	0.3570(− 1)	0.1531(− 1)
0.1	0.4987(− 1)	0.2044(− 1)
0.5	0.2167	0.5957(− 1)
1.0	0.4049	0.8032(− 1)
2.0	0.7659	0.9679(− 1)
3.0	0.1120(+ 1)	0.1027
4.0	0.1472(+ 1)	0.1054
5.0	0.1824(+ 1)	0.1068
7.0	0.2529(+ 1)	0.1081
10.0	0.3587(+ 1)	0.1088
100.0	0.3541(+ 2)	0.1094

TABLE IV. Velocity profiles for $R = 2$.

Radius r	Poiseuille flow Present work	Poiseuille flow Ref. 8	Thermal creep flow Present work	Thermal creep flow Ref. 8
0.000	2.3533	...	0.2970	...
0.004	2.3533	2.3531	0.2970	0.2970
0.026	2.3531	2.3529	0.2970	0.2970
0.070	2.3518	2.3516	0.2969	0.2969
0.135	2.3476	2.3474	0.2966	0.2966
0.219	2.3381	2.3381	0.2959	0.2959
0.321	2.3212	2.3210	0.2946	0.2946
0.437	2.2934	2.2932	0.2925	0.2925
0.567	2.2521	2.2522	0.2893	0.2893
0.706	2.1958	2.1958	0.2848	0.2848
0.851	2.1230	2.1229	0.2788	0.2788
1.000	2.0329	2.0328	0.2710	0.2710
1.149	1.9262	1.9261	0.2613	0.2613
1.294	1.8050	1.8044	0.2495	0.2494
1.433	1.6710	1.6703	0.2354	0.2354
1.563	1.5273	1.5272	0.2190	0.2190
1.679	1.3806	1.3795	0.2006	0.2005
1.781	1.2326	1.2323	0.1802	0.1802
1.865	1.0915	1.0906	0.1585	0.1583
1.930	0.9624	0.9609	0.1364	0.1364
1.974	0.8550	0.8523	0.1160	0.1159
1.996	0.7845	0.7733	0.1011	0.0995

method for evaluating S_0 analytically. In the Appendix, we show how this can be extended to all S_α for even α. This analysis was used successfully as a benchmark for testing the accuracy of the numerical results obtained by Gaussian quadrature.

Next, the linear systems (37) and (38) and (39) and (40) are solved for the F_N coefficients A_α and B_α, $\alpha = 0,1, ..., N$, which are in turn used in Eqs. (47)–(52), (56), and (57) to yield the quantities of interest.

In Table I and Table II the convergence rate of the F_N method is illustrated and the converged results for the flow rates $Q_P(R)$ and $Q_T(R)$ are compared to those of Loyalka,[4] where R is the inverse Knudsen number. We consider the converged results to be correct to ± 1 in the last digit shown. The agreement with Loyalka[8] appears to be best in the Knudsen number range $0.02 < R < 3.0$, although our most accurate results are achieved outside this range. Considered over the complete Knudsen number spectrum, the agreement ranges between three and five significant figures.

Table III contains values for the velocity slip at the wall for both Poiseuille flow and thermal creep flow. To our knowledge these results have not been previously reported. We believe the results to be correct to within ± 1 in the fourth significant figure.

Finally, in Table IV we compare our results for the velocity profiles for $R = 2$ in both Poiseuille flow and thermal creep flow to those of Loyalka.[8] The degree of agreement is quite similar to that found for the flow rates Q_P and Q_T, except very close to the boundary where the agreement drops to one significant figure. Again, we consider our results correct to the number of significant figures shown.

VI. CONCLUSIONS

The F_N method has been used successfully to solve the kinetic theory problems of Poiseuille flow and thermal creep flow in a cylindrical tube. The volumetric flow rates, velocity slip at the wall, and velocity profiles have been computed to an accuracy of at least four significant figures with modest computational effort. Our numerical results spanning the entire range of the Knudsen number indicate that previous work by Cercignani[3] and Loyalka[8] is accurate to within 1%. The present analysis can be used as a benchmark for testing the accuracy of the various numerical methods used previously for the BGK model and in verifying new techniques that might be developed in the future. We are optimistic about extending this work to solving higher order models and models that describe binary flows.

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APPENDIX: ANALYTICAL EVALUATION OF EQUATION (32)

We have mentioned that the integral (32), which can be written as

$$S_\alpha = \int_0^\infty K_1\left(\frac{R}{\mu}\right) I_1\left(\frac{R}{\mu}\right) \mu^\alpha e^{-b\mu^2}\, d\mu, \quad b = 1, \quad \text{(A1)}$$

can be computed analytically, and thus we now proceed to derive this alternative expression. It has been found that the integrals $S_{2\alpha}$, $\alpha = 1,2,3,...$ can be generated by taking the partial derivatives of S_0 with respect to b, through the formula

$$S_{2\alpha} = (-1)^\alpha \frac{\partial^\alpha S_0(R)}{\partial b^\alpha}\bigg|_{b=1}, \quad \alpha = 1,2, \ldots . \quad \text{(A2)}$$

To initiate our calculations we use

$$S_0(R) = R \int_0^\infty K_1\left(\frac{1}{t}\right) I_1\left(\frac{1}{t}\right) e^{-bt^2R^2}\, dt, \quad \text{(A3)}$$

where $t = \mu/R$. A method of evaluating this integral has been pointed out by Ferziger,[5] who gave a three-term asymptotic expansion. We have found the complete result to be given by

$$S_0(b) = \frac{\sqrt{\pi}}{4\sqrt{b}} - \frac{2}{3}R - \frac{\sqrt{\pi}R^2}{8}\left(\ln b + 2\ln R + 3\gamma - \frac{5}{2}\right)\sqrt{b} - \frac{1}{4}\sum_{n=1}^{\infty}\frac{(-1)^n R^{2n+1} b^n}{(n+\frac{3}{2})(n+\frac{1}{2})^2\{(n-\frac{1}{2})!\}^3} - \frac{\sqrt{\pi}}{4}\sum_{n=1}^{\infty}\frac{(-1)^n R^{2n+2} b^{n+\frac{1}{2}}}{(n+2)(n+1)^2(n!)^3} \times\left(\ln b + 2\ln R + 3\gamma - \frac{2}{n+1} - \frac{1}{n+2} - 3\sum_{k=1}^{n}\frac{1}{k}\right). \tag{A4}$$

By setting $b = 1$ we have S_0, and the use of Eq. (A2) yields analytical expressions for the higher-order functions $S_{2\alpha}$, $\alpha = 1,2, \ldots$. The computed exact results of S_α, $\alpha = 2,4, \ldots, 30$, were compared with numerical results obtained directly through a 400 point Gaussian quadrature scheme and agreement was achieved for 9 to 14 significant figures. Based on this success, we used only the numerical method for odd values of α.

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