The Newsvendor model

• **Assumptions/notation**
  - Single-period horizon
  - Uncertain demand in the period: \( D \) (parts) assume continuous random variable
    Density function and cumulative distribution function of \( D \): \( f(x) \) and \( F(x) \)
    \[
    F(a) = P(D \leq a) = \int_{x=0}^{a} f(x)dx \quad f(a) = \frac{dF(x)}{dx}\bigg|_{x=a}
    \]
  - Infinite production/replenishment rate (instantaneous replenishment)
  - Zero lead time
  - Overage cost rate: cost per unit of positive inventory remaining at the end of the period: \( c_o \) (€ per left-over part)
  - Underage cost rate: cost per unit of unsatisfied demand (negative ending inventory): \( c_u \) (€ per missing part or unsatisfied demand)
  - No fixed setup production/order cost

• **Decision**
  - Order quantity at the beginning of the period: \( Q \) (parts)
The Newsvendor model

• Definitions
  – (Positive) inventory remaining at the end of the period: \( I^+ \)
  – Unsatisfied demand (negative inventory) at the end of the period: \( I^- \)
    \[ I^+ = (Q - D)^+ = \max(Q - D, 0) \]
    \[ I^- = (D - Q)^- = \max(D - Q, 0) \]
  – Total overage and underage cost at the end of the period: \( G(Q, D) \)
    \[ G(Q, D) = c_u I^+ + c_o I^- = c_u (Q - D)^+ + c_o (D - Q)^- \]
  – Expected cost: \( G(Q) \)
    \[ G(Q) = E[G(Q, D)] = \int_{x=0}^{\infty} G(Q, x) f(x) dx \]
    \[ = c_u \int_{x=0}^{\infty} (Q - x)^+ f(x) dx + c_o \int_{x=0}^{\infty} (x - Q)^- f(x) dx \]
    \[ = c_u \int_{x=0}^{Q} (Q - x) f(x) dx + c_o \int_{x=Q}^{\infty} (x - Q) f(x) dx \]

\[ \mu = E[D] \]
The Newsvendor model

- Problem
  
  Minimize $G(Q)$

- First derivative of cost function

$$\frac{dG(Q)}{dQ} = \frac{d}{dQ} \left[ c_0 \int_{v=0}^{Q} (Q-x)f(x)dx + c_u \int_{v=Q}^{\infty} (x-Q)f(x)dx \right]$$

  $$= c_u \int_{v=Q}^{\infty} \frac{d}{dQ} (Q-x)f(x)dx + \left[ (Q-x)f(x) \right]_{v=0}^{Q} - 0 \left[ (Q-x)f(x) \right]_{v=Q}^{\infty} +$$

  $$c_u \int_{v=Q}^{\infty} (x-Q)f(x)dx + 0 \left[ (x-Q)f(x) \right]_{v=Q}^{\infty} - 1 \left[ (x-Q)f(x) \right]_{v=Q}^{\infty}$$

  $$= c_u \int_{v=0}^{Q} f(x)dx + c_u \int_{v=Q}^{\infty} (-1)f(x)dx$$

  $$= c_u F(Q) - c_u [1 - F(Q)]$$

- Second derivative

$$\frac{dG^2(Q)}{dQ^2} = \frac{d}{dQ} \left[ c_u F(Q) - c_u [1 - F(Q)] \right] = (c_u + c_u) f(Q) \geq 0$$

- First-order condition for minimization

$$\left. \frac{dG(Q)}{dQ} \right|_{Q=0} = 0 \Rightarrow c_u F(Q) - c_u [1 - F(Q)] = 0 \Rightarrow (c_u + c_u) F(Q) = c_u$$

$$\Rightarrow Q^* : F(Q^*) = \frac{c_u}{c_u + c_u} \Rightarrow Q^* = F^{-1} \left( \frac{c_u}{c_u + c_u} \right)$$

Note:

- $F(Q)$ is the fill rate, i.e. the probability that a demand will be satisfied!

- Recall: EOQ model with backorders: $F^* = \frac{b}{h+b}$
The Newsvendor model

- **Special case:** \( D \sim \text{Normal}(\mu, \sigma) \)

\[
F(Q) = P(D \leq Q) = P\left(\frac{D - \mu}{\sigma} \leq \frac{Q - \mu}{\sigma}\right) = \Phi\left(\frac{Q - \mu}{\sigma}\right)
\]

\[
\Rightarrow F(Q) = \Phi(z), \quad z = \frac{Q - \mu}{\sigma}
\]

\[
\Rightarrow \Phi(z) \text{ and hence } F(Q) \text{ can be evaluated from standardized Normal tables }
\]

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<th>( \Phi(z) - 0.5 )</th>
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<tr>
<td>3.09</td>
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</tbody>
</table>

---

**Example**

\( D \sim \text{Normal}(120, 45) \)

- buying price \( c = 30 \)
- selling price \( S = 110 \)
- salvage price \( s = 10 \)

\[
\Rightarrow \frac{c_v}{c_s + c_v} = \frac{80}{20 + 80} = 0.80 \Rightarrow z_{0.80} = 0.85
\]

\[
\Rightarrow Q' = \mu + \sigma z_{0.80} = 120 + 45 \times 0.85 = 120 + 38.25 = 158.25 \approx 159
\]
The Newsvendor model

• Extension: Discrete demand
  – Uncertain demand in the period: \( D \) (parts) assume discrete random variable
    Probability mass function and cumulative distribution function of \( D \): \( p(x) \) and \( F(x) \)
    \[
    F(a) = P(D \leq a) = \sum_{x=0}^{a} p(x) \quad p(a) = F(a) - F(a-1)
    \]
  – Expected cost: \( G(Q) \)
    \[
    G(Q) = \mathbb{E}_D[Q, D] = \sum_{x=0}^{Q} G(Q, x) p(x) = c_s \sum_{x=0}^{Q-1} (Q-x) p(x) + c_c \sum_{x=Q}^{\infty} (x-Q) p(x)
    \]
  – Problem: Minimize \( G(Q) \)
  – First-order difference
    \[
    G(Q+1) - G(Q) = c_s \sum_{x=0}^{Q} p(x) - c_c \sum_{x=Q+1}^{\infty} p(x) = c_s F(Q) - c_c [1 - F(Q)]
    \]
  – First-order condition for minimization
    \( Q^* \): smallest \( Q \) such that \( G(Q+1) - G(Q) \geq 0 \) \( \Leftrightarrow \) \( c_s F(Q) - c_c [1 - F(Q)] \geq 0 \)
    \[
    \Rightarrow Q^* : \text{smallest } Q \text{ such that } F(Q^*) \geq \frac{c_c}{c_s + c_c}
    \]

The Newsvendor model

• Extension: Staring inventory \( y > 0 \)
  – Still want to be at \( Q^* \) after ordering, because \( Q^* \) is the minimizer of \( G(Q) \)
  – Order quantity: \( U \)
  – Optimal policy now depends on starting inventory:
    \[
    U^*(y) = \begin{cases} 
    Q^* - y, & \text{if } y < Q^* \\
    0, & \text{if } y \geq Q^*
    \end{cases}
    \]

Note:
- \( U^* \) = optimal order quantity
- \( Q^* \) = optimal “order-up-to” point = inventory target level = base stock level
The Newsvendor model

- **Interpretation of** $c_o$ and $c_p$ **for the single-period model**
  - $S =$ selling price (€ per part)
  - $c =$ variable cost (€ per part)
  - $h =$ holding cost (€ per part per period)
  - $p =$ loss-of-goodwill cost (€ per part short per period)

$$G(Q,D) = cQ + h(Q-D)^+ + p(D-Q)^+ - S \min(Q,D)$$

$$G(Q) = E[G(Q,D)] = cQ + h \int_0^Q (Q-x) f(x) dx + p \int_Q^\infty (x-Q) f(x) dx - S \int_0^Q x f(x) dx + \int_0^\infty Q f(x) dx$$

$$\Rightarrow Q^* : F(Q) = \frac{p+S-c}{p+S+h} \Rightarrow c_o = p+S-c, \quad c_p = h+c$$

---

The Newsvendor model

- **Extension: infinite periods (infinite horizon) with backorders**
  
  Same assumptions as single-period model except that:
  - $D_t =$ demand in period $t; D_1, D_2, D_3, \ldots$ are i.i.d. with distribution $f(x), F(x)$
  - $U_t =$ amount ordered in period $t$
  - Optimal policy in each period is “order-up-to” $Q$
  - In the long-run, the inventory can never be higher than $Q$
  
  $\Rightarrow$ In steady-state (long run): $U_t = D_{t-1}$
The Newsvendor model

- **Extension: infinite periods (infinite horizon) with backorders (cont’d)**
  
  - Total cost in a period with demand $D$
    
    $$G(Q, D) = \frac{(c - S)D}{\text{order cost}} + \frac{h(Q - D)^+}{\text{inventory holding cost}} + \frac{p(D - Q)^+}{\text{backorder cost}}$$
    
  - Expected average cost per period
    
    $$G(Q) = E[G(Q, D)] = (c - S)\mu + hE[(Q - D)^+] + pE[(D - Q)^+]$$
    
  - First-order condition for minimizing $G(Q)$
    
    $$\frac{dG(Q)}{dQ} = 0 \implies hF(Q) - p(1 - F(Q)) = 0$$
    
    $$\implies Q': F(Q') = \frac{p}{p + h} \implies \text{Newsventor formula: } c_e = p, \quad c_e = h$$
    
  **Note:**
    
    - $c$ and $S$ play no role in determining $Q'$, because in the long run, all demands are satisfied regardless of $Q$; therefore, the expected average ordering cost minus revenue per period is $(c - S)\mu$ regardless of $Q$.

---

The Newsvendor model

- **Extension: infinite periods (infinite horizon) with lost sales**
  
  Same assumptions as infinite horizon with backorders except that:
  
  - Unmet demand is not backordered but is lost
  
  - In steady-state (long run): $U_t = \min(Q, D_{t-1})$
The Newsvendor model

- **Extension:** infinite periods (infinite horizon) with lost sales (cont’d)
  - Total cost in a period with demand $D$
    \[ G(Q, D) = (c - S)\min(Q, D) + h(D - Q)^+ + p(D - Q)^+ \]
  - Expected average cost per period
    \[ G(Q) = \frac{1}{D} \int G(Q, D) dD = (c - S)\mu + hE[(Q - D)^+] + pE[(D - Q)^+] \]
  - First-order condition for minimizing $G(Q)$
    \[ \frac{dG(Q)}{dQ} = 0 \Rightarrow hF(Q) = (p + S - c)(1 - F(Q)) = 0 \]
    \[ Q^*: F(Q^*) = \frac{p + S - c}{p + S - c + h} \Rightarrow \text{Newsvendor formula: } c_o = p + S - c, \quad \text{h_o} = h \]

**Note:**
- $c$ and $S$ now play a role in determining $Q^*$, because in the long run, the demand satisfied and the orders are $\min(Q, D)$, so they depend on $Q$; therefore, the expected average ordering cost minus revenue per period is $(c - S)\mu\min(Q, D)$.

Lot size – Reorder point ($Q$, $R$) model

- **Assumptions**
  - Infinite horizon
  - Continuous review (as opposed to periodic review)
  - $D_t$: random stationary demand per unit time (e.g., daily demand)
  - $\lambda = E[D_t], \sigma^2 = \text{variance of } D_t$\[ \text{mean } \lambda = E[D_t], \text{variance } \sigma^2 = E[(D_t - \lambda)^2] \]
  - Unmet demand is either backordered or lost
  - $r$: Fixed replenishment order lead time
  - Costs:
    - Variable unit production/order cost: $c$ (€ per part)
    - Fixed setup production/order cost: $K$ (€ per production run/order)
    - Inventory holding cost rate: $h$ (€ per part per unit time)
    - Stock-out (shortage/penalty) cost rate (2 cases / 4 situations: see next)

- **Order policy**
  - $(Q, R)$ policy: order $Q$ when inventory position falls below $R$

- **Decision variables**
  - $Q$: lot size (reorder quantity)
  - $R$: reorder point
\((Q, R)\) model

- **Assumptions on stock-outs and stock-out cost rate**
  - **Case 1: Backordered demand**
    - \(p_1\) (€ per stock-out occasion)
    - \(p_2\) (€ per part short)
    - \(p_3\) (€ per part short per unit time)
  - **Case 2: Lost sales**
    - \(p_L\) (€ per lost sale)

In this course, we only deal with \(p_2\)
\((Q, R)\) model: Backordered demand

- **Analysis**
  
  - **D**: demand during lead time \(\tau\)
  
  - Density function and cumulative distribution function of \(D\): \(f(x)\) and \(F(x)\)
  
  - Mean: \(\mu = E[D] = E[D_1 + D_2 + \ldots + D_\tau] = \tau E[D_1] = \tau \lambda\)
  
  - Variance: \(\sigma^2 = \text{Var}[D] = E[(D - \mu)^2] = \text{Var}[D_1 + D_2 + \ldots + D_\tau] = \tau \text{Var}[D_1] = \tau \sigma^2\)

\[(Q, R)\] model: Backordered demand

- **Inventory holding cost**
  
  - Safety stock \(ss = R - \mu = R - \lambda \tau\)
  
  - Expected average inventory approximation \(\bar{I} \approx ss + \frac{Q}{2} = R - \lambda \tau + \frac{Q}{2}\)
    (underestimates true value)
  
  - Expected average inventory hold cost \(= h \bar{I} = h(R - \lambda \tau + Q/2)\)
(Q, R) model: Backordered demand

- **Setup cost**
  - Expected average order frequency = \( \frac{\lambda}{Q} \)
  - Expected average setup cost per unit time = \( K \frac{\lambda}{Q} \)

- **Stock-out (penalty) cost**
  - Assumption: \( \tau \ll Q/\lambda \) \( \Rightarrow \) stock-out per cycle depends only on \( R \)
  - \( B(R) \): Expected stock-out cost per cycle (depends on definition of stock-out cost rate)
  - Expected average stock-out cost per unit time = \( B(R) \frac{\lambda}{Q} \)

- **Total expected average cost per unit time**

\[
G(Q, R) = h \left( \frac{Q}{2} + R - \lambda \tau \right) + K \frac{\lambda}{Q} + B(R) \frac{\lambda}{Q}
\]

---

(Q, R) model: Backordered demand

- **Optimization problem**

\[
\text{Minimize } G(Q, R) = h \left( \frac{Q}{2} + R - \lambda \tau \right) + K \frac{\lambda}{Q} + B(R) \frac{\lambda}{Q}
\]

- **Optimality conditions**

\[
\frac{\partial G(Q, R)}{\partial Q} = h - \frac{2}{2} K \frac{\lambda}{Q}^2 - \lambda B(R) \frac{\lambda}{Q} = 0 \Rightarrow Q^* = \frac{2 \lambda [K + B(R)]}{h}
\]

\[
\Rightarrow Q = \sqrt{\frac{2 \lambda [K + B(R)]}{h}} \tag{1}
\]

\[
\frac{\partial G(Q, R)}{\partial R} = h + \frac{\lambda}{Q} \frac{dB(R)}{dR} = 0
\]

\[
\Rightarrow \frac{dB(R)}{dR} = -\frac{hQ}{\lambda} \tag{2}
\]
(Q, R) model: Backordered demand

• **Optimality condition (2):** Case 2: Stock-out cost \( p_2 \in \text{per part short} \)

\[
B(R) = p_2 E[(D - R)^+] = p_2 \begin{array}{c}
\int_{x=R}^{\infty} (x - R) f(x) dx
\end{array} \Rightarrow \frac{dB(R)}{dR} = -p_2 [1 - F(R)]
\]

Condition (2): \(-p_2 [1 - F(R)] = -\frac{hQ}{\lambda} \Rightarrow F(R) = 1 - \frac{hQ}{p_2 \lambda}\)

---

(Q, R) model: Backordered demand

• **Simultaneous solution of conditions (1) and (2)**

Solve by fixed-point iteration:
1. Given \( R \), solve (1) to find \( Q \)
2. Given \( Q \), solve (2) to find \( R \)
3. Repeat until convergence
(Q, R) model: Backordered demand

Illustration for case 2: Stock-out cost $p_2$ € per part short

- Optimality conditions for case 2

\[
Q = \sqrt{\frac{2\lambda(K + p_2 n(R))}{h}} \quad (1) \quad F(R) = 1 - \frac{hQ}{p_2 \lambda} \quad (2)
\]

- Assumption: $D \sim \text{Normal}(\mu, \sigma)$

Use standardized cumulative distribution function (cdf) $\Phi(z)$ to compute $F(R)$

\[
F(R) = P(D \leq R) = P\left( \frac{D - \mu}{\sigma} \leq \frac{R - \mu}{\sigma} \right) = \Phi\left( \frac{R - \mu}{\sigma} \right)
\]

\[
\Rightarrow F(R) = \Phi(z), \quad z = \frac{R - \mu}{\sigma}
\]

- $\Phi(z)$ and hence $F(R)$ can be evaluated from standardized Normal cdf tables

(Q, R) model: Backordered demand

Use standardized loss function $L(z)$ to compute $n(R)$

\[
Y \sim \text{Normal}(0,1) \quad \Rightarrow \quad L(z) = E[(Y - z)^+] = \int_{z^-}^{+\infty} (y - z) \phi(y) \, dy
\]

\[
n(R) = E[(D - R)^+] = E\left[ \sigma \left( \frac{D - \mu}{\sigma} - \frac{R - \mu}{\sigma} \right) \right] = \sigma L\left( \frac{R - \mu}{\sigma} \right)
\]

\[
\Rightarrow n(R) = \sigma L(z), \quad z = \frac{R - \mu}{\sigma}
\]

$L(z)$ and hence $n(R)$ can be evaluated from standardized loss function tables.

It can be shown that

\[
L(z) = \phi(z) - z[1 - \Phi(z)]
\]

\[
\Rightarrow n(R) = \sigma L(z) = \sigma \phi(z) + (\mu - R)[1 - \Phi(z)], \quad z = \frac{R - \mu}{\sigma}
\]
(Q, R) model: Backordered demand

Under the assumption $D \sim \text{Normal}(\mu, \sigma)$, the optimality conditions become

\[
Q = \sqrt{\frac{2\lambda[K + p_2\sigma L(z)]}{h}} \quad (1)
\]

\[
\Phi(z) = 1 - \frac{Qh}{p_1\lambda} \quad (2)
\]

\[
z = \frac{R - \mu}{\sigma} \quad (3)
\]

(Q, R) model: Backordered demand

Fixed point iteration algorithm for case 2 under the assumption $D \sim \text{Normal}(\mu, \sigma)$

\[
Q_0 = \sqrt{\frac{2\lambda K}{h}}, \quad z_0 = \phi^{-1}\left(1 - \frac{Q_0h}{p_1\lambda}\right), \quad R_0 = \mu + \sigma z_0, \quad n = 1
\]

Step 1: \[
Q_n = \sqrt{\frac{2\lambda[K + p_2\sigma L(z_{n-1})]}{h}}
\]

Step 2: \[
z_n = \phi^{-1}\left(1 - \frac{Q_n h}{p_1\lambda}\right)
\]

Step 3: \[
R_n = \mu + \sigma z_n
\]

Step 4: \[
|Q_n - Q_{n-1}| \geq \epsilon \quad \text{or} \quad |R_n - R_{n-1}| \geq \epsilon \quad \Rightarrow \quad n \leftarrow n + 1, \quad \text{GOTO Step 1}
\]
(Q, R) model: Service Levels

- **Service levels in (Q, R) systems**
  - **Type 1 Service** (replaces stock-out cost $p_1$ € per stock-out occasion)
    - $S_1 =$ Probability of not stocking out during the lead time
    - $S_1 = P(D \leq R) = F(R)$
  - **Optimization problem**
    - Minimize $G(Q, R) = h\left(\frac{Q}{2} + R - \lambda \tau\right) + K \frac{\lambda}{Q}$
    - subject to $F(R) \geq \alpha$ (i.e., subject to $S_1 \geq \alpha$)
  - **Solution**
    - $Q^* = \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}$
    - $R^* =$ minimum $R$ such that $F(R) \geq \alpha$
    - $D$ continuous r.v. $\Rightarrow R^* = F^{-1}(\alpha)$

- **Type 2 Service** (replaces stock-out cost $p_2$ € per part short)
  - $S_2 =$ Proportion of demands met from stock
  - $S_2 = 1 - \frac{n(R)}{Q}$
  - **Optimization problem**
    - Minimize $G(Q, R) = h\left(\frac{Q}{2} + R - \lambda \tau\right) + K \frac{\lambda}{Q}$
    - subject to $1 - \frac{n(R)}{Q} \geq \beta$ (i.e., subject to $S_2 \geq \beta$)
  - **Note:** Now the constraint depends on both $R$ and $Q$
(Q, R) model: Service Levels

Type 2 Service (cont’d)

Approximate solution

\[ Q' \approx \sqrt{\frac{2K\lambda}{h}} = \text{EOQ} \]

\( R' \) = minimum \( R \) such that \( n(R) \leq Q'(1 - \beta) \)

\( D \) continuous r.v. \[ n(R') = Q'(1 - \beta) \]

\( D \sim \text{Normal}(\mu, \sigma) \Rightarrow n(R') = \sigma L(z') = Q'(1 - \beta) \)

\[ R' = \mu + \sigma z' \quad z' = L^{-1}\left(\frac{Q'(1 - \beta)}{\sigma}\right) \]

---

More accurate solution

Consider first-order conditions (1) and (2) for case 2

\[ Q = \sqrt{\frac{2\lambda[K + p_z n(R)]}{h}} \quad (1), \quad F(R) = 1 - \frac{Qh}{p_z \lambda} \quad (2) \]

(2) \[ p_z = \frac{Qh}{[1 - F(R)]\lambda} \] = imputed stock-out cost

(1) \[ Q = \sqrt{\frac{2\lambda[K + Qh n(R)]/[1 - F(R)]}{h}} \] is quadratic function in \( Q \)

positive root:

\[ Q = \frac{n(R)}{1 - F(R)} \sqrt{\frac{2K\lambda}{h} + \left(\frac{n(R)}{1 - F(R)}\right)^2} \quad (3) \]

\( n(R) = (1 - \beta)Q \) \[ (4) \Rightarrow L(z) = \frac{(1 - \beta)Q}{\sigma} \]
\((Q, R)\) model: Random Lead Time

**Extension: Random lead-time**

- **\(L\):** random lead time
- **Mean:** \(r = E[L]\), variance \(\sigma_L^2 = E[(L- \tau)^2]\)
- **\(D\):** demand during lead time \(L\)
  - Density function and cumulative distribution function of \(D\): \(f(x)\) and \(F(x)\)
  - \(D = D_1 + D_2 + \ldots + D_L\), where \(L\) is a random variable
  - It can be shown (see next page) that:
    - Mean: \(\mu = E[D] = \tau \lambda\)
    - Variance: \(\sigma^2 = Var[D] = E[(D - \mu)^2] = \tau \sigma_L^2 + \lambda^2 \sigma_T^2\)

Everything else holds!!

---

\((Q, R)\) model: Random Lead Time

**Derivation of \(\mu\) and \(\sigma^2\)**

Mean: \(\mu = E[D] = E[ E[D|L]] = E[ E[DL]] = \tau \lambda\)

Variance: \(\sigma^2 = Var[D] = E[(D - \mu)^2] = E[D^2] - 2 \mu D + \mu^2 = E[D^2] - 2 \mu E[D] + E[\mu^2]\)

\[
= E[ E[D^2|L]] - 2 \mu^2 + \mu^2 = \tau \sigma_T^2 + \lambda^2 \sigma_L^2 + \lambda^2 r^2 - \mu^2 = \tau \sigma_T^2 + \lambda^2 \sigma_L^2 + \lambda^2 r^2 - \tau^2 \lambda^2
\]

\[
= \tau \sigma_T^2 + \lambda^2 \sigma_L^2
\]

where we used:

\[
E[D^2|L] = E[Var[D|L] + E[D|L]^2] = L \sigma_T^2 + L \lambda^2
\]

\[
E[ E[D^2|L]] = E( L \sigma_T^2 + L \lambda^2) = \tau \sigma_T^2 + \lambda^2 (Var[L] + r^2) = \tau \sigma_T^2 + \lambda^2 (\sigma_L^2 + r^2) = \tau \sigma_T^2 + \lambda^2 \sigma_L^2 + \lambda^2 r^2
\]