Stochastic (Random) Demand Inventory Models

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The Newsvendor model

• Assumptions/notation
  – Single-period horizon
  – Uncertain demand in the period: $D$ (parts) assume continuous random variable
    Density function and cumulative distribution function of $D$: $f(x)$ and $F(x)$
    \[ F(a) = P(D \leq a) = \int_{x=0}^{a} f(x)dx \quad f(a) = \left. \frac{dF(x)}{dx} \right|_{x=a} \]
  – Infinite production/replenishment rate (instantaneous replenishment)
  – Zero lead time
  – Overage cost rate: cost per unit of positive inventory remaining at the end of
    the period: $c_o$ (€ per left-over part)
  – Underage cost rate: cost per unit of unsatisfied demand (negative ending
    inventory) : $c_u$ (€ per missing part or unsatisfied demand)
  – No fixed setup production/order cost

• Decision
  – Order quantity at the beginning of the period: $Q$ (parts)
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• Definitions
  – (Positive) inventory remaining at the end of the period: $I^+$
  – Unsatisfied demand (negative inventory) at the end of the period: $I^-$
    \[ I^+ = (Q - D)^+ = \max(Q - D, 0) \]
    \[ I^- = (D - Q)^+ = \max(D - Q, 0) \]
  – Total overage and underage cost at the end of the period: $G(Q, D)$
    \[ G(Q, D) = c_o I^+ + c_u I^- = c_o (Q - D)^+ + c_u (D - Q)^+ \]
  – Expected cost: $G(Q)$
    \[ G(Q) = \mathbb{E}_D[G(Q, D)] = \int_0^\infty G(Q, x) f(x) dx \]
    \[ = c_o \int_0^\infty (Q - x)^+ f(x) dx + c_u \int_0^\infty (x - Q)^+ f(x) dx \]
    \[ = c_o \int_0^Q (Q - x) f(x) dx + c_u \int_Q^\infty (x - Q) f(x) dx \]
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\[ I^+ = (Q - D)^+ \equiv \max(Q - D, 0) \]

\[ I^- = (D - Q)^+ \equiv \max(D - Q, 0) \]

\[ c_o \int_{x=0}^{Q} (Q - x) f(x) dx \]

\[ \mu = E[D] \]

\[ Q \]

\[ D \]
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• Problem

Minimize $G(Q)$

• First derivative of cost function

\[
\frac{dG(Q)}{dQ} = \frac{d}{dQ} \left[ c_o \int_{x=0}^{Q} (Q-x)f(x)dx + c_u \int_{x=Q}^{\infty} (x-Q)f(x)dx \right]
\]

\[
= c_o \int_{x=0}^{Q} \frac{d}{dQ} (Q-x)f(x)dx + 1(Q-x)f(x)|_{x=Q} - 0(Q-x)f(x)|_{x=0} +
\]

\[
c_u \int_{x=Q}^{\infty} \frac{d}{dQ} (x-Q)f(x)dx + 0(x-Q)f(x)|_{x=\infty} - 1(x-Q)f(x)|_{x=Q}
\]

\[
= c_o \int_{x=0}^{Q} f(x)dx + c_u \int_{x=Q}^{\infty} (-1)f(x)dx
\]

\[
= c_o F(Q) - c_u [1 - F(Q)]
\]
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• In deriving the previous formula, we used Leibnitz’s rule for taking the derivative of an integral whose limits are functions of the variable with respect to which the derivative is taken. Leibnitz’s rule is:

\[
\frac{d}{dy} \int_{a(y)}^{b(y)} f(x, y) dx = \int_{a(y)}^{b(y)} \frac{df(x, y)}{dy} dx + f(b(y), y) \frac{db(y)}{dy} - f(a(y), y) \frac{da(y)}{dy}
\]
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- Second derivative

\[
\frac{dG^2(Q)}{dQ^2} = \frac{d}{dQ} \left[ c_o F(Q) - c_u [1 - F(Q)] \right] = (c_o + c_u) f(Q) \geq 0
\]

- First-order condition for minimization

\[
\frac{dG(Q)}{dQ} \bigg|_{Q=0} = 0 \quad \Rightarrow \quad c_o F(Q) - c_u [1 - F(Q)] = 0 \quad \Rightarrow \quad (c_o + c_u) F(Q) = c_u
\]

\[
\Rightarrow \quad Q^* : F(Q^*) = \frac{c_u}{c_o + c_u} \quad \Rightarrow \quad Q^* = F^{-1} \left( \frac{c_u}{c_o + c_u} \right)
\]

Note:

- \( F(Q) \) is the fill rate, i.e. the probability that a demand will be satisfied!

- Recall: EOQ model with backorders: \( F^* = \frac{b}{h + b} \)
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- **Special case:** $D \sim \text{Normal}(\mu, \sigma)$

$$F(Q) = P(D \leq Q) = P\left(\frac{D - \mu}{\sigma} \leq \frac{Q - \mu}{\sigma}\right) = \Phi\left(\frac{Q - \mu}{\sigma}\right)$$

$\Rightarrow F(Q) = \Phi(z), \quad z = \frac{Q - \mu}{\sigma}$

$\Rightarrow \Phi(z)$ and hence $F(Q)$ can be evaluated from standardized Normal tables

<table>
<thead>
<tr>
<th>$z$</th>
<th>$\Phi(z)$</th>
<th>$\Phi(z) - 0.5$</th>
</tr>
</thead>
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<tr>
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<td>0.0000</td>
</tr>
<tr>
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<td>0.8023</td>
<td>0.3023</td>
</tr>
<tr>
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<td>0.9015</td>
<td>0.4015</td>
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<tr>
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<td>0.9505</td>
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</tr>
<tr>
<td>2.33</td>
<td>0.9901</td>
<td>0.4901</td>
</tr>
<tr>
<td>3.09</td>
<td>0.9990</td>
<td>0.4990</td>
</tr>
</tbody>
</table>
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- **Special case:** $D \sim \text{Normal}(\mu, \sigma)$ cont’d

\[ Q^* : F(Q^*) = \frac{c_u}{c_o + c_u} \Rightarrow \Phi \left( \frac{Q^* - \mu}{\sigma} \right) = \frac{c_u}{c_o + c_u} \Rightarrow \frac{Q^* - \mu}{\sigma} = \Phi^{-1} \left( \frac{c_u}{c_o + c_u} \right) \]

\[ \Rightarrow Q^* = \mu + \sigma z \frac{c_u}{c_o + c_u} \]

- **Example**

$D \sim \text{Normal}(120, 45)$

- Buying price $c = 30$
- Selling price $S = 110$
- Salvage price $s = 10$

\[ c_u = S - c = 110 - 30 = 80 \]
\[ c_o = c - s = 30 - 10 = 20 \]

\[ \Rightarrow \frac{c_u}{c_o + c_u} = \frac{80}{20 + 80} = 0.80 \Rightarrow z_{0.80} = 0.85 \]

\[ \Rightarrow Q^* = \mu + \sigma z_{0.80} = 120 + 45 \cdot 0.85 = 120 + 38.25 = 158.25 \approx 159 \]
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• Extension: Discrete demand
  
  – Uncertain demand in the period: $D$ (parts) assume discrete random variable
    
    Probability mass function and cumulative distribution function of $D$: $p(x)$ and $F(x)$
    
    $$F(a) = P(D \leq a) = \sum_{x \leq a} p(x) \quad p(a) = F(a) - F(a - 1)$$
  
  – Expected cost: $G(Q)$
    
    $$G(Q) = E[G(Q, D)] = \sum_{x} G(Q, x) p(x) = c_o \sum_{x=0}^{Q-1} (Q - x) p(x) + c_u \sum_{x=Q}^{\infty} (x - Q) p(x)$$
  
  – Problem: Minimize $G(Q)$
    
  – First-order difference
    
    $$G(Q + 1) - G(Q) = c_o \sum_{x=0}^{Q} p(x) - c_u \sum_{x=Q+1}^{\infty} p(x) = c_o F(Q) - c_u [1 - F(Q)]$$
  
  – First-order condition for minimization
    
    $$Q^*: \text{smallest } Q \text{ such that } G(Q + 1) - G(Q) \geq 0 \iff c_o F(Q) - c_u [1 - F(Q)] \geq 0$$
    
    $$\Rightarrow Q^*: \text{smallest } Q \text{ such that } F(Q^*) \geq \frac{c_u}{c_o + c_u}$$
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- **Extension: Staring inventory** \( y > 0 \)
  - Still want to be at \( Q^* \) after ordering, because \( Q^* \) is the minimizer of \( G(Q) \)
  - Order quantity: \( U \)
  - Optimal policy now depends on starting inventory:

\[
U^*(y) = \begin{cases} 
Q^* - y, & \text{if } y < Q^* \\
0, & \text{if } y \geq Q^*
\end{cases}
\]

**Note:**
- \( U^* \equiv \text{optimal order quantity} \)
- \( Q^* \equiv \text{optimal “order-up-to” point } \equiv \text{inventory target level } \equiv \text{base stock level} \)
The Newsvendor model

- **Interpretation of** $c_o$ **and** $c_u$ **for the single-period model**
  - $S = \text{selling price (€ per part)}$
  - $c = \text{variable cost (€ per part)}$
  - $h = \text{holding cost (€ per part per period)}$
  - $p = \text{loss-of-goodwill cost (€ per part short per period)}$

$$G(Q,D) = cQ + h(Q-D)^+ + p(D-Q)^+ - S \min(Q,D)$$

$$G(Q) = E[G(Q,D)] = cQ + h \int_0^Q (Q-x)f(x)dx + p \int_Q^\infty (x-Q)f(x)dx - S[\int_0^Q xf(x)dx + \int_Q^{\infty} Qf(x)dx]$$

$$= cQ + h \int_0^Q (Q-x)f(x)dx + (p+S) \int_Q^\infty (x-Q)f(x)dx - S\mu$$

$$\frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad c + hF(Q) - (p+S)(1-F(Q)) = 0$$

$$\Rightarrow \quad Q^* : F(Q) = \frac{p+S-c}{p+S+h} \quad \Rightarrow \quad c_u = p+S-c, \quad c_o = h+c$$
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- **Extension: infinite periods (infinite horizon) with backorders**

  Same assumptions as single-period model except that:
  
  - $D_t = \text{demand in period } t$; $D_1, D_2, D_3, \ldots$ are i.i.d. with distribution $f(x)$, $F(x)$
  - $U_t = \text{amount ordered in period } t$
  - Optimal policy in each period is “order-up-to” $Q$
  - In the long-run, the inventory can never be higher than $Q$

  $\Rightarrow$ In steady-state (long run): $U_t = D_{t-1}$
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• Extension: infinite periods (infinite horizon) with backorders (cont’d)

– Total cost in a period with demand $D$

$$G(Q, D) = (c - S)D + h(Q - D)^+ + p(D - Q)^+$$

– Expected average cost per period

$$G(Q) = E[D G(Q, D)] = (c - S)\mu + hE[(Q - D)^+] + pE[(D - Q)^+]$$

– First-order condition for minimizing $G(Q)$

$$\frac{dG(Q)}{dQ} = 0 \implies hF(Q) - p(1 - F(Q)) = 0$$

$$\implies Q^*: F(Q^*) = \frac{p}{p + h} \implies \text{Newsventor formula: } c_u = p, \quad c_o = h$$

Note:

– $c$ and $S$ play no role in determining $Q^*$, because in the long run, all demands are satisfied regardless of $Q$; therefore, the expected average ordering cost minus revenue per period is $(c - S)\mu$ regardless of $Q$. 

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The Newsvendor model

- **Extension: infinite periods (infinite horizon) with lost sales**

  Same assumptions as infinite horizon with backorders except that:
  
  - Unmet demand is not backordered but is lost
  - In steady-state (long run): \( U_t = \min(Q, D_{t-1}) \)
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- Extension: infinite periods (infinite horizon) with lost sales (cont’d)
  - Total cost in a period with demand $D$
    $$G(Q, D) = (c - S)\min(Q, D) + h(Q - D)^+ + p(D - Q)^+$$
  - Expected average cost per period
    $$G(Q) = \frac{1}{D}G(Q, D) = (c - S)\left[\mu - E[(D - Q)^+]\right] + hE[(Q - D)^+] + pE[(D - Q)^+]$$
  - First-order condition for minimizing $G(Q)$
    $$\frac{dG(Q)}{dQ} = 0 \quad \Rightarrow \quad hF(Q) - (p + S - c)(1 - F(Q)) = 0$$
    $$\Rightarrow \quad Q^*: F(Q^*) = \frac{p + S - c}{p + S - c + h} \quad \Rightarrow \quad \text{Newsvendor formula: } c_u = p + S - c, \quad c_o = h$$

Note:
- $c$ and $S$ now play a role in determining $Q^*$, because in the long run, the demand satisfied and the orders are $\min(Q, D)$, so they depend on $Q$; therefore, the expected average ordering cost minus revenue per period is $(c - S)E[\min(Q, D)]$. 
Lot size – Reorder point \((Q, R)\) model

- **Assumptions**
  - Infinite horizon
  - Continuous review (as opposed to periodic review)
  - \(D_t\): random stationary demand per unit time (e.g., daily demand)
    
    \[
    \lambda \equiv E[D_t], \text{ variance } \sigma_t^2 = E[(D_t - \lambda)^2]
    \]
  - Unmet demand is either backordered or lost
  - \(\tau\): Fixed replenishment order lead time
  - Costs:
    - Variable unit production/order cost: \(c\) (€ per part)
    - Fixed setup production/order cost: \(K\) (€ per production run/order)
    - Inventory holding cost rate: \(h\) (€ per part per unit time)
    - Stock-out (shortage/penalty) cost rate (2 cases / 4 situations: see next)

- **Order policy**
  - \((Q, R)\) policy: order \(Q\) when *inventory position* falls below \(R\)

- **Decision variables**
  - \(Q\): lot size (reorder quantity)
  - \(R\): reorder point
(Q, R) model

• Assumptions on stock-outs and stock-out cost rate
  
  – Case 1: Backordered demand
    • $p_1$ (€ per stock-out occasion)
    • $p_2$ (€ per part short )
    • $p_3$ (€ per part short per unit time)
  
  – Case 2: Lost sales
    • $p_L$ (€ per lost sale)
(\(Q, R\)) model: Backordered demand

Inventory position

On-hand inventory

Backorders

\(R+Q\)

\(R\)

\(Q\)

\(\tau\)

time
(Q, R) model: Backordered demand

- **Analysis**
  - D: demand during lead time $\tau$
    - Density function and cumulative distribution function of $D$: $f(x)$ and $F(x)$
    - $D = D_1 + D_2 + \ldots + D_\tau$
    - Mean: $\mu \equiv E[D] = E[D_1 + D_2 + \ldots + D_\tau] = \tau E[D_t] = \tau \lambda$
    - Variance: $\sigma^2 \equiv Var[D] = E[(D - \mu)^2] = Var[D_1 + D_2 + \ldots + D_\tau] = \tau Var[D_t] = \tau \sigma_t^2$
(Q, R) model: Backordered demand

• Inventory holding cost
  – Safety stock \( ss \equiv R - \mu = R - \lambda \tau \)
  – Expected average inventory approximation \( \bar{I} \approx ss + Q/2 = R - \lambda \tau + Q/2 \) (underestimates true value)
  – Expected average inventory hold cost = \( h \bar{I} = h(R - \lambda \tau + Q/2) \)
(Q, R) model: Backordered demand

To see why expected average inventory approximation underestimates true expected average inventory:

\[ \overline{I}_{\text{appr}} \approx R - E[D] + Q/2 = R - \int_0^\infty x f(x)dx + Q/2 \]

\[ \overline{I}_{\text{true}} = E[(R - D)^+] + Q/2 = \int_0^R (R - x) f(x)dx + Q/2 = RF(R) + \int_0^R x f(x)dx + Q/2 \]

\[ \overline{I}_{\text{appr}} - \overline{I}_{\text{true}} = R(1 - F(R)) - \int_0^\infty x f(x)dx \]

\[ = R \int_0^\infty f(x)dx - \int_0^\infty x f(x)dx \]

\[ = \int_0^\infty R f(x)dx - \int_0^\infty x f(x)dx \]

\[ = \int_0^\infty (R - x) f(x)dx \]

\[ = E[(D - R)^+] \leq 0 \]

\[ \Rightarrow \overline{I}_{\text{appr}} \leq \overline{I}_{\text{true}} \]
\((Q, R)\) model: Backordered demand

- **Setup cost**
  - Expected average order frequency = \(\lambda/Q\)
  - Expected average setup cost per unit time = \(K \lambda/Q\)

- **Stock-out (penalty) cost**
  - Assumption: \(\tau \ll Q/\lambda\) ⇒ stock-out per cycle depends only on \(R\)
  - \(B(R):\) Expected stock-out cost per cycle (depends on definition of stock-out cost rate)
  - Expected average stock-out cost per unit time = \(B(R) \lambda/Q\)

- **Total expected average cost per unit time**
  \[
  G(Q, R) = h\left(\frac{Q}{2} + R - \lambda \tau\right) + K \frac{\lambda}{Q} + B(R) \frac{\lambda}{Q}
  \]
(\(Q, R\)) model: Backordered demand

- **Optimization problem**

\[
\text{Minimize } G(Q, R) = h\left(\frac{Q}{2} + R - \lambda \tau\right) + K\frac{\lambda}{Q} + \frac{\lambda}{Q}B(R)
\]

- **Optimality conditions**

\[
\frac{\partial G(Q, R)}{\partial Q} = h - \frac{K\lambda}{2Q^2} - \frac{\lambda B(R)}{Q^2} = 0 \quad \Rightarrow \quad Q^2 = \frac{2\lambda[K + B(R)]}{h}
\]

\[\Rightarrow \quad Q = \sqrt{\frac{2\lambda[K + B(R)]}{h}} \quad (1)\]

\[
\frac{\partial G(Q, R)}{\partial R} = h + \frac{\lambda}{Q} dB(R) = 0
\]

\[\Rightarrow \quad \frac{dB(R)}{dR} = -\frac{hQ}{\lambda} \quad (2)\]
(Q, R) model: Backordered demand

- **Optimality condition (2): 3 cases**
  - **Case 1**: Stock-out cost $p_1 \€$ per stock-out occasion
    \[
    B(R) = p_1 [1 - F(R)] \quad \Rightarrow \quad \frac{dB(R)}{dR} = -p_1 f(R)
    \]
    Condition (2): $-p_1 f(R) = -\frac{hQ}{\lambda} \quad \Rightarrow \quad f(R) = \frac{hQ}{p_1 \lambda}$
  - **Case 2**: Stock-out cost $p_2 \€$ per part short
    \[
    B(R) = p_2 \mathbb{E}[(D - R)^+] = p_2 \int_{x=R}^{\infty} (x - R) f(x)dx \quad \Rightarrow \quad \frac{dB(R)}{dR} = -p_2 [1 - F(R)]
    \]
    Condition (2): $-p_2 [1 - F(R)] = -\frac{hQ}{\lambda} \quad \Rightarrow \quad F(R) = 1 - \frac{hQ}{p_2 \lambda}$
(Q, R) model: Backordered demand

- Case 3: Stock-out cost $p_3 \text{ € per part short per unit time}$

Average waiting time per backorder when the demand is $D$:

$$\frac{(D - R)^+}{2\lambda}$$

Average total waiting time of all backorders when the demand is $D$:

$$\frac{(D - R)^+}{2\lambda} = \left[\frac{(D - R)^+}{2\lambda}\right]^2$$

$$B(R) = p_3 \int_{x=R}^{\infty} f(x)dx = \frac{p_3}{2\lambda} \int_{x=R}^{\infty} (x - R)^2 f(x)dx$$

$$dB(R) = \frac{p_3}{\lambda} \int_{x=R}^{\infty} (x - R) f(x)dx = -\frac{p_3}{\lambda} E[(D - R)^+] = -\frac{p_3}{\lambda} n(R)$$

Condition (2): \(-\frac{p_3}{\lambda} n(R) = -\frac{hQ}{\lambda} \Rightarrow n(R) = \frac{hQ}{p_3}\)
$(Q, R)$ model: Backordered demand

- **Simultaneous solution of conditions (1) and (2)**
  
  Solve by fixed-point iteration:
  
  1. Given $R$, solve (1) to find $Q$
  2. Given $Q$, solve (2) to find $R$
  3. Repeat until convergence
(Q, R) model: Backordered demand

Illustration for case 2: Stock-out cost $p_2 \in \text{ per part short}$

- Optimality conditions for case 2

\[
Q = \sqrt{\frac{2\lambda[K + p_2 n(R)]}{h}} \quad (1)
\]

\[
F(R) = 1 - \frac{hQ}{p_2 \lambda} \quad (2)
\]

- Assumption: $D \sim \text{Normal}(\mu, \sigma)$

Use standardized cumulative distribution function (cdf) $\Phi(z)$ to compute $F(R)$

\[
F(R) = P(D \leq R) = P \left( \frac{D - \mu}{\sigma} \leq \frac{R - \mu}{\sigma} \right) = \Phi \left( \frac{R - \mu}{\sigma} \right)
\]

\[
\Rightarrow \quad F(R) = \Phi(z), \quad z = \frac{R - \mu}{\sigma}
\]

- $\Phi(z)$ and hence $F(R)$ can be evaluated from standardized Normal cdf tables
(Q, R) model: Backordered demand

Use standardized loss function $L(z)$ to compute $n(R)$

$$Y \sim \text{Normal}(0,1) \quad \Rightarrow \quad L(z) \equiv E[(Y - z)^+] = \int_{y=-\infty}^{\infty} (y - z) \varphi(y) \, dy$$

$$n(R) = E[(D - R)^+] = E \left[ \sigma \left( \frac{D - \mu}{\sigma} - \frac{R - \mu}{\sigma} \right)^+ \right] = \sigma L \left( \frac{R - \mu}{\sigma} \right)$$

$$\Rightarrow \quad n(R) = \sigma L(z), \quad z = \frac{R - \mu}{\sigma}$$

$L(z)$ and hence $n(R)$ can be evaluated from standardized loss function tables

It can be shown that

$$L(z) = \varphi(z) - z[1 - \Phi(z)]$$

$$\Rightarrow \quad n(R) = \sigma L(z) = \sigma \varphi(z) + (\mu - R)[1 - \Phi(z)], \quad z = \frac{R - \mu}{\sigma}$$
\((Q, R)\) model: Backordered demand

Under the assumption \(D \sim \text{Normal}(\mu, \sigma)\), the optimality conditions become

\[
Q = \sqrt{\frac{2\lambda[K + p_2\sigma L(z)]}{h}} \tag{1}
\]

\[
\Phi(z) = 1 - \frac{Qh}{p_2\lambda} \tag{2}
\]

\[
z = \frac{R - \mu}{\sigma} \tag{3}
\]
(Q, R) model: Backordered demand

Fixed point iteration algorithm for case 2 under the assumption $D \sim \text{Normal}(\mu, \sigma)$

$$Q_0 = \sqrt{\frac{2\lambda K}{h}}, \quad z_0 = \Phi^{-1}\left(1 - \frac{Q_0 h}{p\lambda}\right), \quad R_0 = \mu + \sigma z_0, \quad n = 1$$

Step 1: $$Q_n = \sqrt{\frac{2\lambda[K + p_2 \sigma L(z_{n-1})]}{h}}$$

Step 2: $$z_n = \Phi^{-1}\left(1 - \frac{Q_n h}{p\lambda}\right)$$

Step 3: $$R_n = \mu + \sigma z_n$$

Step 4: $$|Q_n - Q_{n-1}| \geq \varepsilon \quad \text{OR} \quad |R_n - R_{n-1}| \geq \varepsilon \quad \Rightarrow \quad n \leftarrow n + 1, \quad \text{GOTO Step 1}$$
(Q, R) model: Lost sales

Inventory position

On-hand inventory

Lost sales

τ
(Q, R) model: Lost sales

Modify total expected average cost per unit time function

\[ G(Q, R) = h \left( \frac{Q}{2} + R - \lambda \tau + n(R) \right) + K \frac{\lambda}{Q} + p_L \frac{\lambda}{Q} n(R) \]

Optimality conditions

\[ Q = \frac{\sqrt{2\lambda[K + p_L n(R)]}}{h} \quad (1) \quad \text{(same as case 2 of backordered demands)} \]

\[ \frac{\partial G(Q, R)}{\partial R} = h \left( 1 + \frac{dn(R)}{dR} \right) + p_L \frac{\lambda}{Q} \frac{dn(R)}{dR} = 0 \quad \Rightarrow \quad 1 + \left( 1 + \frac{p_L \lambda}{Q h} \right) \frac{dn(R)}{dR} = 0 \]

\[ 1 - \left( 1 + \frac{p_L \lambda}{Q h} \right) \left[ 1 - F(R) \right] = 0 \]

\[ F(R) = 1 - \frac{Q h}{Q h + p_L \lambda} = \frac{p_L \lambda}{Q h + p_L \lambda} \quad (2) \quad \text{(Newsvendor formula: c_o = Qh, c_u = p_L \lambda)} \]

\[ (Q h \ll p_L \lambda \quad \Rightarrow \quad \text{same as case 2 of backordered demands}) \]
(Q, R) model: Lost sales

Note

- G(Q, R) is an approximation, because it overestimates the order frequency.
- In the lost sales model, not all demand is met; on average, Q demands are met and n(R) demands are not met.

⇒ Average number of demands per cycle = Q + n(R) parts per cycle.
- Number of demands per unit time = λ parts per unit time.
⇒ Order frequency = λ/[Q + n(R)] instead of λ/Q.
⇒ A more accurate expression for G(Q, R) is:

\[
G(Q, R) = h \left( \frac{Q}{2} + R - \lambda \tau + n(R) \right) + K \frac{\lambda}{Q + n(R)} + p_L \frac{\lambda}{Q + n(R)} n(R)
\]
- Usually, n(R) \ll Q, so the order frequency can still be approximated by \( \lambda/Q \).
(\(Q, R\)) model: Summary

\[ Q = \sqrt{\frac{2\lambda[K + B(R)]}{h}} \]
\[ z = \frac{R - \mu}{\sigma} \]

<table>
<thead>
<tr>
<th>Situation</th>
<th>(B(R))</th>
<th>(B(R)) D~Normal((\mu, \sigma))</th>
<th>Eq. (2)</th>
<th>Eq. (2) D~Normal((\mu, \sigma))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_1) (€ per stock-out occasion)</td>
<td>(p_1[1 - F(R)])</td>
<td>(p_1[1 - \Phi(z)])</td>
<td>(f(R) = \frac{hQ}{p_1\lambda})</td>
<td>(\varphi(z) = \frac{\sigma hQ}{p_1\lambda})</td>
</tr>
<tr>
<td>(p_2) (€ per part short)</td>
<td>(p_2n(R))</td>
<td>(p_2\sigma L(z))</td>
<td>(F(R) = 1 - \frac{hQ}{p_2\lambda})</td>
<td>(\Phi(z) = 1 - \frac{hQ}{p_2\lambda})</td>
</tr>
<tr>
<td>(p_3) (€ per part short per unit time)</td>
<td>(\frac{p_3}{2\lambda} \int_{x=R}^{\infty} (x - R)^2 f(x)dx)</td>
<td></td>
<td>(n(R) = \frac{hQ}{p_3})</td>
<td>(L(z) = \frac{hQ}{\sigma p_3})</td>
</tr>
<tr>
<td>(p_L) (€ per lost sale)</td>
<td>(p_Ln(R))</td>
<td>(p_L\sigma L(z))</td>
<td>(F(R) = \frac{p_L\lambda}{Qh + p_L\lambda})</td>
<td>(\Phi(z) = \frac{p_L\lambda}{Qh + p_L\lambda})</td>
</tr>
</tbody>
</table>
(Q, R) model: Service Levels

- Service levels in (Q, R) systems
  - Type 1 Service (replaces stock-out cost $p_1 \, \text{€ per stock-out occasion}$)
    
    $S_1 \equiv$ Probability of not stocking out during the lead time
    
    $S_1 = P(D \leq R) = F(R)$
  
  - Optimization problem

    Minimize $G(Q,R) = h\left(\frac{Q}{2} + R - \lambda \tau\right) + K \frac{\lambda}{Q}$

    subject to $F(R) \geq \alpha$ (i.e., subject to $S_1 \geq \alpha$)

- Solution

  $Q^* = \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}$

  $R^* = \text{minimum } R \text{ such that } F(R) \geq \alpha$

  $D$ continuous r.v. $\Rightarrow \quad R^* = F^{-1}(\alpha)$
(Q, R) model: Service Levels

- **Type 2 Service** (replaces stock-out cost \( p_2 \) € per part short)
  \[ S_2 \equiv \text{Proportion of demands met from stock} \]
  \[ S_2 = 1 - \frac{n(R)}{Q} \]

- **Optimization problem**

  \[
  \text{Minimize } G(Q, R) = h\left(\frac{Q}{2} + R - \lambda \tau \right) + K \frac{\lambda}{Q}
  \]

  subject to \( 1 - \frac{n(R)}{Q} \geq \beta \) (i.e., subject to \( S_2 \geq \beta \))

- **Note**: Now the constraint depends on both \( R \) and \( Q \)
(\(Q, R\)) model: Service Levels

- Type 2 Service (cont’d)

Approximate solution

\[
Q^* \approx \sqrt{\frac{2K\lambda}{h}} = \text{EOQ}
\]

\(R^* = \text{minimum } R \text{ such that } n(R) \leq Q^* (1 - \beta)\)

\(D \text{ continuous r.v. } \Rightarrow n(R^*) = Q^* (1 - \beta)\)

\(D \sim \text{Normal}(\mu, \sigma) \Rightarrow n(R^*) = \sigma L(z^*) = Q^* (1 - \beta)\)

\[R^* = \mu + \sigma z^*, \quad z^* = L^{-1}\left(\frac{Q^* (1 - \beta)}{\sigma}\right)\]
(Q, R) model: Service Levels

Type 2 Service (cont’d)

More accurate solution

Consider first-order conditions (1) and (2) for case 2

\[ Q = \sqrt{\frac{2\lambda[K + pn(R)]}{h}} \] \hspace{1cm} (1), \hspace{1cm} F(R) = 1 - \frac{Qh}{p\lambda} \hspace{1cm} (2)

\[ (2) \Rightarrow p = \frac{Qh}{[1 - F(R)]\lambda} \equiv \text{imputed stock-out cost} \]

\[ (1) \Rightarrow Q = \sqrt{\frac{2\lambda[K + Qhn(R)]/[1 - F(R)]\lambda}{h}} \equiv \text{quadratic function in } Q \]

positive root: \[ Q = \frac{n(R)}{1 - F(R)} + \sqrt{\frac{2K\lambda}{h} + \left(\frac{n(R)}{1 - F(R)}\right)^2} \] \hspace{1cm} (3)

\[ n(R) = \frac{(1 - \beta)Q}{\sigma} \] \hspace{1cm} (4)
(Q, R) model: Random Lead Time

- **Extension:** Random lead-time
  - \( L \): random lead time
  - Mean: \( \tau \equiv E[L] \), variance \( \sigma_L^2 \equiv E[(L - \tau)^2] \)
  - \( D \): demand during lead time \( L \)
    - Density function and cumulative distribution function of \( D \): \( f(x) \) and \( F(x) \)
    - \( D = D_1 + D_2 + \ldots + D_L \), where \( L \) is a random variable
    - It can be shown (see next page) that:
      - Mean: \( \mu \equiv E[D] = \tau \lambda \)
      - Variance: \( \sigma^2 \equiv Var[D] = E[(D - \mu)^2] = \tau \sigma_i^2 + \lambda^2 \sigma_L^2 \)

Everything else holds!!
\((Q, R)\) model: Random Lead Time

- Derivation of \(\mu\) and \(\sigma^2\)

Mean: \(\mu \equiv E[D] = E[L \cdot E[D | L]] = E[L \lambda] = \tau \lambda\)

Variance: \(\sigma^2 \equiv Var[D] = E[(D - \mu)^2] = E[D^2 - 2 \mu D + \mu^2] = E[D^2] - 2 \mu E[D] + E[\mu^2] = E[L \cdot E[D^2 | L]] - 2 \mu^2 + \mu^2 = \tau \sigma_i^2 + \lambda^2 \sigma_L^2 + \lambda^2 \tau^2 - \mu^2 = \tau \sigma_i^2 + \lambda^2 \sigma_L^2 + \lambda^2 \tau^2 - \tau^2 \lambda^2 = \tau \sigma_i^2 + \lambda^2 \sigma_L^2\)

where we used:

\[E[D^2 | L] = E[D^2] = E[Var[D | L] + E[D | L]^2] = L \sigma_i^2 + L^2 \lambda^2\]

\[E[L \cdot E[D^2 | L]] = E[L \sigma_i^2 + L^2 \lambda^2] = \tau \sigma_i^2 + \lambda^2 (Var[L] + \tau^2) = \tau \sigma_i^2 + \lambda^2 (\sigma_L^2 + \tau^2) = \tau \sigma_i^2 + \lambda^2 \sigma_L^2 + \lambda^2 \tau^2\]