Chapter Three

Aggregate Planning

Chapter Overview

Purpose
To develop techniques for aggregating units of production, and determining appropriate aggregate units of production and workforce levels based on predicted demand for aggregate units.

Key Points
1. Aggregate units of production. This chapter could also have been called Macro Production Planning, since the purpose of aggregating units is to be able to develop a top-down plan for the entire firm or for some subset of the firm, such as a product line or a particular plant. For large firms producing a wide range of products or for firms providing a service rather than a product, determining appropriate aggregate units can be a challenge. The most direct approach is to express aggregate units in some generic measure, such as dollars of sales, tons of steel, or gallons of paint. For a service, such as provided by a consulting firm or a law firm, billed hours would be a reasonable way of expressing aggregate units.

2. Aspects of aggregate planning. The following are the most important features of aggregate planning:
   - Smoothing. Costs that arise from changing production and workforce levels.
   - Bottlenecks. Planning in anticipation of peak demand periods.
   - Planning horizon. One must choose the number of periods considered carefully. If too short, sudden changes in demand cannot be anticipated. If too long, demand forecasts become unreliable.
   - Treatment of demand. All the mathematical models in this chapter consider demand to be known, i.e., have zero forecast error.

3. Costs in aggregate planning.
   - Smoothing costs. The cost of changing production and/or workforce levels.
   - Holding costs. The opportunity cost of dollars invested in inventory.
   - Shortage costs. The costs associated with back-ordered or lost demand.
   - Labor costs. These include direct labor costs on regular time, overtime, subcontracting costs, and idle time costs.

4. Solving aggregate planning problems. Approximate solutions to aggregate planning problems can be found graphically, and exact solutions via linear programming. When solving problems graphically, the first step is to draw a graph of the cumulative net demand curve. If the goal is to develop a level plan (i.e., one that has constant production or workforce levels over the planning horizon), then one matches the cumulative net demand curve as closely as possible with a straight line. If the goal is to develop a zero-inventory plan (i.e., one that minimizes holding and shortage costs), then one tracks the cumulative net demand curve as closely as possible each period. While linear programming provides cost optimal solutions, the method does not take into account management policy, such as avoiding hiring and firing as much as possible. For a problem with a 7 period planning horizon, the linear programming formulation requires 8T variables and 3T constraints. For long planning horizons, this can become quite tedious. Another issue that must be dealt with is that the solution to a linear program is noninteger. To handle this problem, one would either have to specify that the problem variables were integers (which could make the problem computationally unwieldy) or develop some suitable rounding procedure.

5. The linear decision rule. The aggregate planning concept had its roots in the work of Holt, Modigliani, Muth, and Simon (1960) who developed a model for Pittsburgh Paints (presumably) to determine their workforce and production levels. The model used quadratic approximations for the costs, and obtained simple linear equations for the optimal policies. This work spawned the later interest in aggregate planning.

6. Modeling management behavior. Bowman (1963) considered linear decision rules similar to those derived by Holt, Modigliani, Muth, and Simon except that he suggested fitting the parameters of the model based on management's actions, rather than describing optimal actions based on cost minimization. This is one of the few examples of a mathematical model used to describe human behavior in the context of operations planning.

7. Disaggregating aggregate plans. While aggregate planning is useful for providing approximate solutions for macro planning at the firm level, the question is whether these aggregate plans provide any guidance for planning at the lower levels of the firm. A disaggregation scheme is a means of taking an aggregate plan and breaking it down to get more detailed plans at lower levels of the firm.

As we go through life, we make both micro and macro decisions. Micro decisions might be what to eat for breakfast, what route to take to work, what auto service to use, or which movie to rent. Macro decisions are the kind that change the course of one's life: where to live, what to major in, which job to take, whom to marry. A company also must make both micro and macro decisions every day. In this chapter we explore decisions made at the macro level, such as planning companywide workforce and production levels.

Aggregate planning, which might also be called macro production planning, addresses the problem of deciding how many employees the firm should retain and, for a manufacturing firm, the quantity and the mix of products to be produced. Macro planning is not limited to manufacturing firms. Service organizations must determine employee staffing needs as well. For example, airlines must plan staffing levels for flight attendants and pilots, and hospitals must plan staffing levels for nurses. Macro planning strategies are a fundamental part of the firm's overall business strategy. Some firms operate on the philosophy that costs can be controlled only by making frequent changes in the size and/or composition of the workforce. The aerospace industry in California in the 1970s adopted this strategy. As government contracts shifted from one producer to another, so
Chapter Three

Aggregate Planning

did the technical workforce. Other firms have a reputation for retaining employees, even in bad times. Until recently, IBM and AT&T were two well-known examples.

A manufacturer who provides a service or produces a product, macro planning begins with the forecast of demand. Techniques for demand forecasting were presented in Chapter 2. How responsive the firm can be to anticipated changes in the demand depends on several factors. These factors include the general strategy the firm may have regarding retaining workers and its commitments to existing employees. As noted in Chapter 2, demand forecasts are generally wrong because there is almost always a random component of the demand that cannot be predicted exactly in advance. The aggregate planning methodology discussed in this chapter requires that demand is deterministic, or known in advance. This assumption is made to simplify the analysis and allow us to focus on the systematic or predictable changes in the demand pattern, rather than on the unsystematic or random changes. Inventory management subject to randomness is treated in detail in Chapter 5.

Traditionally, most manufacturers have chosen to retain primary production in-house. Some components might be purchased from outside suppliers (see the discussion of the make-or-buy problem in Chapter 1), but the primary product is traditionally produced by the firm. Henry Ford was one of the first American manufacturers to design a completely vertically integrated business. Ford even owned a stand of rubber trees so it would not have to purchase rubber for tires.

That philosophy is undergoing a dramatic change, however. In dynamic environments, firms are finding that they can be more flexible if the manufacturing is outsourced; that is, if it is done on a subcontract basis. One example is Sun Microsystems, a California-based producer of computer workstations. Sun, a market leader, adopted the strategy of focusing on product innovation and design rather than on manufacturing. It has developed close ties to contract manufacturers such as San Jose-based Soleciton Corporation, winner of the Baldrige Award for Quality. Subcontracting its primary manufacturing function has allowed Sun to be more flexible and to focus on innovation in a rapidly changing market.

Aggregate planning involves competing objectives. One objective is to react quickly to anticipated changes in demand, which would require making frequent and potentially large changes in the size of the labor force. Such a strategy has been called a chase strategy. This may be cost effective, but could be a poor long-run business strategy. Workers who are laid off may not be available when business turns around. For this reason, the firm may wish to adopt the objective of retaining a stable workforce. However, this strategy who are laid off may not be available when business turns around. For this reason, the firm may wish to adopt the objective of retaining a stable workforce. However, this strategy would be more appropriate if many different types of items are produced, it would be more appropriate to consider aggregate units in terms of weight (tons of steel), volume (gallons of gasoline), amount of work required (worker-years of programming time), or dollar value (value of inventory in dollars). What the appropriate aggregating scheme should be is not always obvious. It depends on the context of the particular planning problem and the level of aggregation required.

Example 3.1

A plant manager working for a large national appliance firm is considering implementing an aggregate planning system to determine the workforce and production levels in his plant. This particular plant produces six models of washing machines. The characteristics of the machines are:

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Required to Produce</th>
<th>Selling Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A5532</td>
<td>4.2</td>
<td>$285</td>
</tr>
<tr>
<td>K4242</td>
<td>4.9</td>
<td>$345</td>
</tr>
<tr>
<td>L9898</td>
<td>5.1</td>
<td>$395</td>
</tr>
<tr>
<td>L3800</td>
<td>5.2</td>
<td>$425</td>
</tr>
<tr>
<td>M2624</td>
<td>5.4</td>
<td>$525</td>
</tr>
<tr>
<td>M3988</td>
<td>5.8</td>
<td>$725</td>
</tr>
</tbody>
</table>

The plant manager must decide on the particular aggregation scheme to use. One possibility is to define an aggregate unit as one dollar of output. Unfortunately, the selling prices of the various models of washing machines are not consistent with the number of worker-hours required to produce them. The ratio of the selling price divided by the worker-hours is $67.18 for A5532 and $125.00 for M3880. The company bases its pricing on the fact that the less expensive models have a higher sales volume. The manager notices that the percentages of the total number of sales for these six models have been fairly constant, with values of 32 percent for A5532, 17 percent for K4242, 16 percent for L9898, 12 percent for L3800, 10 percent for M2624, and 6 percent for M3988. He decides to define an aggregate unit of production as a fictitious washing machine requiring (3.24X2) + (2.45X4) + (1.75X1) + (1.45X2) + (1.95X4) + (0.65X8) = 8.856 hours of labor time. He can obtain sales forecasts for each specific production unit in essentially the same way by multiplying the appropriate fractions by the forecasts for unit sales of each type of machine.

The approach used by the plant manager in Example 3.1 was possible because of the relative similarity of the products produced. However, defining an aggregate unit of production at a higher level of the firm is more difficult. In cases in which the firm produces a large variety of products, a natural aggregate unit is sales dollars. Although, as we saw in the example, this will not necessarily translate to the same number of units of production for each item, it will generally provide a good approximation for planning at the highest level of a firm that produces a diverse product line.
3.2 Overview of the Aggregate Planning Problem

The goal of aggregate planning is to determine aggregate production quantities and the levels of resources required to achieve these production goals. In practice, this translates to finding the number of workers that should be employed and the number of aggregate units to be produced in each of the planning periods $1, 2, \ldots, T$. The objective of aggregate planning is to balance the advantages of producing to meet demand as closely as possible against the disruptions caused by changing the levels of production and/or the workforce levels.

The primary issues related to the aggregate planning problem include:

1. **Smoothing.** Smoothing refers to costs that result from changing production and workforce levels from one period to the next. Two of the key components of smoothing costs are the costs that result from hiring and firing workers. Aggregate planning methodology requires the specification of these costs, which may be difficult to estimate. Hiring workers could have far-reaching consequences and costs that may be difficult to evaluate. Firms that hire and fire frequently develop a poor public image. This could adversely affect sales and discourage potential employees from joining the company. Furthermore, workers that are laid off might not simply wait around for business to pick up. Firing workers can have a detrimental effect on the future size of the labor force if those workers obtain employment in other industries. Finally, most companies are simply not at liberty to hire and fire at will. Labor agreements restrict the freedom of management to freely alter workforce levels. However, it is still valuable for management to be aware of the cost trade-offs associated with varying workforce levels and the attendant savings in inventory costs.

2. **Bottleneck problems.** We use the term bottleneck to refer to the inability of the system to respond to sudden changes in demand as a result of capacity restrictions. For example, a bottleneck could arise when the forecast for demand in one month is unusually high, and the plant does not have sufficient capacity to meet that demand. A breakdown of a vital piece of equipment also could result in a bottleneck. The primary issues related to the aggregate planning problem include:

3. **Planning horizon.** The number of periods for which the demand is to be forecasted, and hence the number of periods for which workforce and inventory levels are to be determined, must be specified in advance. The choice of the value of the forecast horizon, $T$, can be significant in determining the usefulness of the aggregate plan. If $T$ is too small, then current production levels might not be adequate for meeting the demand beyond the horizon length. If $T$ is too large, it is likely that the forecasts far into the future will prove inaccurate. If future demands turn out to be very different from the forecasts, then current decisions indicated by the aggregate plan could be incorrect. Another issue involving the planning horizon is the end-of-horizon effect. For example, the aggregate plan might recommend that the inventory at the end of the horizon be drawn to zero in order to minimize holding costs. This could be a poor strategy, especially if demand increases at that time. (However, this particular problem can be avoided by adding a constraint specifying minimum ending inventory levels.)

In practice, rolling schedules are almost always used. This means that at the time of the next decision, a new forecast of demand is appended to the former forecasts and old forecasts might be revised to reflect new information. The new aggregate plan may recommend different production and workforce levels for the current period than were recommended one period ago. When only the decisions for the current planning period need to be implemented immediately, the schedule should be viewed as dynamic rather than static.
Although rolling schedules are common, it is possible that because of production lead times, the schedule must be frozen for a certain number of planning periods. This means that decisions over some collection of future periods cannot be altered. The most direct means of dealing with frozen horizons is simply to label as period 1 the first period in which decisions are not frozen.

4. Treatment of demand. As noted above, aggregate planning methodology requires the assumption that demand is known with certainty. This is simultaneously a weakness and a strength of the approach. It is a weakness because it ignores the possibility (and, in fact, likelihood) of forecast errors. As noted in the discussion of forecasting techniques in Chapter 2, it is virtually a certainty that demand forecasts are wrong. Aggregate planning does not provide any buffer against unanticipated forecast errors. However, most inventory models that allow for random demand require that the average demand be constant over time. Aggregate planning allows the manager to focus on the systematic changes that are generally not present in models that assume random demand. By assuming deterministic demand, the effects of seasonal fluctuations and business cycles can be incorporated into the planning function.

3.3 COSTS IN AGGREGATE PLANNING

As with most of the optimization problems considered in production management, the goal of the analysis is to choose the aggregate plan that minimizes cost. It is important to identify and measure those specific costs that are affected by the planning decision.

1. Smoothing costs. Smoothing costs are those costs that accrue as a result of changing the production levels from one period to the next. In the aggregate planning context, the most salient smoothing cost is the cost of changing the size of the workforce. Increasing the size of the workforce requires time and expense to advertise positions, interview prospective employees, and train new hires. Decreasing the size of the workforce means that workers must be laid off. Severance pay is thus one cost of decreasing the size of the workforce. Other costs, somewhat harder to measure, are (a) the costs of a decline in worker morale that may result and (b) the potential for decreasing the size of the labor pool in the future, as workers who are laid off acquire jobs with other firms or in other industries.

Most of the models that we consider assume that the costs of increasing and decreasing the size of the workforce are linear functions of the number of employees that are hired or fired. That is, there is a constant dollar amount charged for each employee hired or fired. The assumption of linearity is probably reasonable up to a point. As the supply of labor becomes scarce, there may be additional costs required to hire more workers, and the costs of laying off workers may go up substantially if the number of workers laid off is too large. A typical cost function for changing the size of the workforce appears in Figure 3-2.

2. Holding costs. Holding costs are the costs that accrue as a result of having capital tied up in inventory. If the firm can decrease its inventory, the money saved could be invested elsewhere with a return that will vary with the industry and with the specific company. (A more complete discussion of holding costs is deferred to Chapter 4.) Holding costs are almost always assumed to be linear in the number of units being held at a particular point in time. We will assume for the purposes of the aggregate planning analysis that the holding cost is expressed in terms of dollars per unit held per planning period. We also will assume that holding costs are charged against the inventory remaining on hand at the end of the planning period. This assumption is made for convenience only. Holding costs could be charged against starting inventory or average inventory as well.

3. Shortage costs. Holding costs are charged against the aggregate inventory as long as it is positive. In some situations it may be necessary to incur shortages, which are represented by a negative level of inventory. Shortages can occur when forecasted demand exceeds the capacity of the production facility or when demands are higher than anticipated. For the purposes of aggregate planning, it is generally assumed that excess demand is backlogged and filled in a future period. In a highly competitive situation, however, it is possible that excess demand is lost and the customer goes elsewhere. This case, which is known as lost sales, is more appropriate in the management of single items and is more common in a retail than in a manufacturing context.

As with holding costs, shortage costs are generally assumed to be linear. Convex functions also can accurately describe shortage costs, but linear functions seem to be the most common. Figure 3-3 shows a typical holding/shortage cost function.

4. Regular time costs. These costs involve the cost of producing one unit of output during regular working hours. Included in this category are the actual payroll costs of regular employees working on regular time, the direct and indirect costs of materials, and other manufacturing expenses. When all production is carried out on regular time, regular payroll costs become a "sunk cost," because the number of units produced must equal the number of units demanded over any planning horizon of sufficient length. If there is no overtime or worker idle time, regular payroll costs do not have to be included in the evaluation of different strategies.

5. Overtime and subcontracting costs. Overtime and subcontracting costs are the costs of producing units not produced on regular time. Overtime refers to production by regular-time employees beyond the normal workday, and subcontracting refers to the production of items by an outside supplier. Again, it is generally assumed that both of these costs are linear.

6. Idle time costs. The complete formulation of the aggregate planning problem also includes a cost for underutilization of the workforce, or idle time. In most contexts,
FIGURE 3-3
Holding and back-order costs

![Diagram showing holding and back-order costs with slope equations and inventory levels.]

the idle time cost is zero, as the direct costs of idle time would be taken into account in
labor costs and lower production levels. However, idle time could have other con-
sequences for the firm. For example, if the aggregate units are input to another process,
idle time on the line could result in higher costs to the subsequent process. In such
cases, one would explicitly include a positive idle cost.

When planning is done at a relatively high level of the firm, the effects of intangible
factors are more pronounced. Any solution to the aggregate planning problem obtained
from a cost-based model must be considered carefully in the context of company pol-
icy. An optimal solution to a mathematical model might result in a policy that requires
frequent hiring and firing of personnel. Such a policy may be infeasible because of
prior contract agreements, or undesirable because of the potential negative effects on
the firm’s public image.

Problems for Sections 3.1–3.3

1. What does the term aggregate unit of production mean? Do aggregate production
units always correspond to actual items? Do they ever? Discuss.

2. What is the aggregation scheme recommended by Hax and Meal?

3. Discuss the following terms and their relationship to the aggregate planning
problem:
   a. Smoothing
   b. Bottlenecks
   c. Capacity
   d. Planning horizon

4. A local machine shop employs 60 workers who have a variety of skills. The shop
accepts one-time orders and also maintains a number of regular clients. Discuss some
of the difficulties with using the aggregate planning methodology in this context.

5. A large manufacturer of household consumer goods is considering integrating an
aggregate planning model into its manufacturing strategy. Two of the company
vice presidents disagree strongly as to the value of the approach. What arguments
might each of the vice presidents use to support his or her point of view?

6. Describe the following costs and discuss the difficulties that arise in attempting to
measure them in a real operating environment.
   a. Smoothing costs
   b. Holding costs
   c. Payroll costs

7. Discuss the following statement: “Since we use a rolling production schedule, I
really don’t need to know the demand beyond next month.”

8. St. Clair County Hospital is attempting to assess its needs for nurses over the com-
 ing four months (January to April). The need for nurses depends on both the
numbers and the types of patients in the hospital. Based on a study conducted by
consultants, the hospital has determined that the following ratios of nurses to
patients are required:

<table>
<thead>
<tr>
<th>Patient Type</th>
<th>Numbers of Nurses Required per Patient</th>
<th>Patient Forecasts</th>
</tr>
</thead>
<tbody>
<tr>
<td>Major surgery</td>
<td>0.4</td>
<td>28</td>
</tr>
<tr>
<td>Minor surgery</td>
<td>0.1</td>
<td>12</td>
</tr>
<tr>
<td>Maternity</td>
<td>0.5</td>
<td>22</td>
</tr>
<tr>
<td>Critical care</td>
<td>0.6</td>
<td>75</td>
</tr>
<tr>
<td>Other</td>
<td>0.3</td>
<td>80</td>
</tr>
</tbody>
</table>

   a. How many nurses should be working each month to most closely match patient
      forecasts?

   b. Suppose the hospital does not want to change its policy of not increasing the
      nursing staff size by more than 10 percent in any month. Suggest a schedule of
      nurse staffing over the four months that meets this requirement and also meets
      the need for nurses each month.

3.4 A PROTOTYPE PROBLEM

One can obtain adequate solutions for many aggregate planning problems by hand or by
using relatively straightforward graphical techniques. Linear programming is a means
of obtaining (nearly) optimal solutions. We illustrate the different solution techniques
with the following example.

Example 3.2

Densepack is to plan workforce and production levels for the six-month period January to June.
The firm produces a line of disk drives for mainframe computers that are plug compatible with
several computers produced by major manufacturers. Forecast demands over the next six
months for a particular line of drives produced in the Milpitas, California, plant are 1,280, 640,
900, 1,200, 2,000, and 1,400. There are currently (end of December) 300 workers employed
in the Milpitas plant. Ending inventory in December is expected to be 500 units, and the firm
would like to have 690 units on hand at the end of June.
There are several ways to incorporate the starting and the ending inventory constraints into the formulation. The most convenient is simply to modify the values of the predicted demand. Define net predicted demand in period 1 as the predicted demand minus initial inventory. If there is a minimum ending inventory constraint, then this amount should be added to the demand in period 1. Minimum buffer inventories also can be handled by modifying the predicted demand. If there is a minimum buffer inventory in every period, this amount should be added to the first period's demand. If there is a minimum buffer inventory in only one period, this amount should be added to that period's demand and subtracted from the next period's demand. Actual ending inventories should be computed using the original demand pattern, however.

Returning to our example, we define the net predicted demand for January as $780 = (1,280 - 500)$ and the net predicted demand for June as $2,000 = (1,400 + 600)$. By considering net demand, we may make the simplifying assumption that starting and ending inventories are both zero. The net predicted demand and the net cumulative demand for the six months January to June are as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Net Predicted Demand</th>
<th>Net Cumulative Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>780</td>
<td>780</td>
</tr>
<tr>
<td>February</td>
<td>640</td>
<td>1,420</td>
</tr>
<tr>
<td>March</td>
<td>900</td>
<td>2,320</td>
</tr>
<tr>
<td>April</td>
<td>2,200</td>
<td>3,520</td>
</tr>
<tr>
<td>May</td>
<td>2,000</td>
<td>5,520</td>
</tr>
<tr>
<td>June</td>
<td>2,000</td>
<td>7,520</td>
</tr>
</tbody>
</table>

The cumulative net demand is pictured in Figure 3-4. A production plan is the specification of the production levels for each month. If shortages are not permitted, then cumulative production must be at least as great as cumulative demand each period. In addition to the cumulative net demand, Figure 3-4 also shows one feasible production plan.

In order to illustrate the cost trade-offs of various production plans, we will assume in the example that there are only three costs to be considered: cost of hiring workers, cost of firing workers, and cost of holding inventory. Define:

\[ c_h = \text{Cost of hiring one worker} = \$500, \]
\[ c_f = \text{Cost of firing one worker} = \$2,000, \]
\[ c_i = \text{Cost of holding one unit of inventory for one month} = \$80. \]

We require a means of translating aggregate production in units to workforce levels. Because not all months have an equal number of working days, we will use a day as an indivisible unit of measure and define

\[ K = \text{Number of aggregate units produced by one worker in one day}. \]

In the past, the plant manager observed that over 22 working days, with the workforce level constant at 76 workers, the firm produced 245 disk drives. That means that on average the production rate was $245/22 = 11.164$ drives per day when there were 76 workers employed at the plant. It follows that one worker produced an average of $11.164/76 = 0.14653$ drive per day. Hence, $K = 0.14653$ for this example.

We will evaluate two alternative plans for managing the workforce that represent two essentially opposite management strategies. Plan 1 is to change the workforce each month in order to produce enough units to most closely match the demand pattern. This is known as a zero inventory plan. Plan 2 is to maintain the minimum constant workforce necessary to satisfy the net demand. This is known as the constant workforce plan.

**Evaluation of a Chase Strategy (Zero Inventory Plan)**

Here, we will develop a production plan for Densepack that minimizes the levels of inventory the firm must hold during the six-month planning horizon. Table 3-1 summarizes the input information for the calculations and shows the minimum number of workers required in each month.

One obtains the entries in the final column of Table 3-1, the minimum number of workers required each month, by dividing the forecasted net demand by the number of units produced per worker. The value of this ratio is then rounded upward to the next whole number.

<table>
<thead>
<tr>
<th>TABLE 3-1</th>
<th>Initial Calculations for Zero Inventory Plan for Densepack</th>
</tr>
</thead>
<tbody>
<tr>
<td>Month</td>
<td>Number of Working Days</td>
</tr>
<tr>
<td>January</td>
<td>20</td>
</tr>
<tr>
<td>February</td>
<td>24</td>
</tr>
<tr>
<td>March</td>
<td>18</td>
</tr>
<tr>
<td>April</td>
<td>26</td>
</tr>
<tr>
<td>May</td>
<td>32</td>
</tr>
<tr>
<td>June</td>
<td>15</td>
</tr>
</tbody>
</table>

In conclusion, the chase strategy minimizes inventory, whereas the constant workforce plan minimizes the number of workers. Both strategies have their drawbacks and advantages.
higher integer. We must round upward to guarantee that shortages do not occur. As an example, consider the month of January. Forming the ratio $780 / 2.931$ gives 266.12, which rounded to 267 workers. The number of working days each month depends on a variety of factors, such as paid holidays and worker schedules. The reduced number of days in June is due to a planned shutdown of the plant in the last week of June.

Recall that the number of workers employed at the end of December is 300. Hiring and firing workers each month to match forecast demand as closely as possible results in the aggregate plan given in Table 3-2.

The number of units produced each month (column F in Table 3-2) is obtained by the following formula:

$$\text{Number of units produced per month} = \frac{\text{Number of workers} \times \text{Number of units produced per worker}}{\text{Average number of workers}}$$

rounded to the nearest integer.

The total cost of this production plan is obtained by multiplying the totals at the bottom of Table 3-2 by the appropriate costs. For example, the total cost of hiring, firing, and holding is $(255)(500) + (145)(1,000) + (30)(80) = 524,900$. This cost must now be adjusted to include the cost of holding for the ending inventory of 600 units, which was netted out of the demand for January. Hence, the total cost of this plan is $524,900 + (600)(80) = 572,900$. Note that the initial inventory of 500 units does not enter into the calculations because it will be netted out during the month of January.

It is usually impossible to achieve zero inventory at the end of each planning period because it is not possible to employ a fractional number of workers. For this reason, there will almost always be some inventory remaining at the end of each period in addition to the inventory required to be on hand at the end of the planning horizon. It is possible that ending inventory in one or more periods could build up to a point where the size of the workforce could be reduced by one or more workers. In this example there is sufficient inventory on hand to reduce the workforce by one worker in the months of both March and May. Check that the resulting plan hires a total of 753 workers and fires a total of 144 workers and has a total of only 13 units of inventory, the cost of this modified plan comes to $569,540.

**Evaluation of the Constant Workforce Plan**

Now assume that the goal is to eliminate completely the need for hiring and firing during the planning horizon. In order to guarantee that shortages do not occur in any period, it is necessary to compute the minimum workforce required for every month in the planning horizon. For January, the net cumulative demand is 780 and there are 2,931 units produced per worker, resulting in a minimum workforce of 267 in January. There are exactly $2,931 + 3,517 = 6,448$ units produced per worker in January and February combined, which have a cumulative demand of 1,420. Hence, 1,420/6,448 = 0.2222 = 222 workers are required to cover both January and February. Continuing to form the ratios of the cumulative net demand and the cumulative number of units produced per worker for each month in the horizon results in Table 3-3.

The minimum number of workers required for the entire six-month planning period is the maximum entry in column D in Table 3-3, which is 411 workers. It is only a coincidence that the maximum ratio occurred in the final period.

Because there are 300 workers employed at the end of December, the constant workforce plan requires hiring 111 workers at the beginning of January. No further hiring and firing of workers are required. The inventory levels that result from a constant workforce of 411 workers appear in Table 3-4. The monthly production levels in column C of the table are obtained by multiplying the number of units produced per worker each month by the fixed workforce size of 411 workers. The total of the ending inventory levels is 5,962 + 600 = 6,562. (Recall the 600 units that were netted out of the demand for January.) Hence the total inventory cost of this plan is (5,962)(80) = $524,960. To this we add the cost of increasing the workforce from 300 to 411 in January, which is (111)(500) = $55,500, giving a total cost of this plan of $580,460. This is somewhat higher than the cost of the zero inventory plan, which was $569,540. However, because costs of the two plan are close, it is likely that the company would prefer the constant workforce plan in order to avoid any unaccounted for costs of making frequent changes in the workforce.
Mixed Strategies and Additional Constraints

The zero inventory plan and constant workforce strategies just treated are pure strategies: they are designed to achieve one objective. With more flexibility, small modifications can result in dramatically lower costs. One might question the interest in manual calculations considering that aggregate planning problems can be formulated and solved optimally by linear programming.

Manual calculations enhance intuition and understanding. Computers are dumb. It is easy to overlook a critical constraint or objective when using a computer. It is important to have a feel for the right solution before solving a problem on a computer, so that glaring mistakes are obvious. An important skill, largely ignored these days, is being able to do a ballpark calculation in one's head before pulling out the calculator or computer.

Figure 3-4 shows the constant workforce strategy for Densepack. The hatched area represents the inventory carried in each month. Suppose we allow a single change in the production rate during the six months. Can you identify a strategy from the figure that substantially reduces inventory without permitting shortages?

Graphically, the problem is to cover the cumulative net demand curve with two straight lines, rather than one straight line. This can be accomplished by driving the net inventory to zero at the end of period 4 (April). To do so, we need to produce enough in each of the months January through April to meet the cumulative net demand each month. That means we need to produce $3,520/4 = 880$ units in each of the first four months. In Figure 3-4, the line connecting the origin to the cumulative net demand in April lies wholly above the cumulative net demand curve for the prior months. If the graph is accurate, that means that there should be no shortages occurring in these months. The May and June production is then set to $2,000$, exactly matching the net demand in these months. With this policy we obtain

<table>
<thead>
<tr>
<th>Month</th>
<th>Cumulative Net Demand</th>
<th>Cumulative Production</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>780</td>
<td>880</td>
</tr>
<tr>
<td>February</td>
<td>1,420</td>
<td>1,760</td>
</tr>
<tr>
<td>March</td>
<td>2,320</td>
<td>2,640</td>
</tr>
<tr>
<td>April</td>
<td>3,520</td>
<td>3,520</td>
</tr>
<tr>
<td>May</td>
<td>5,520</td>
<td>5,520</td>
</tr>
<tr>
<td>June</td>
<td>7,520</td>
<td>7,520</td>
</tr>
</tbody>
</table>

As we will see in Section 3.5, this policy turns out to be optimal for the Densepack problem.

The graphical solution method also can be used when additional constraints are present. For example, suppose that the production capacity of the plant is only 1,800 units per month. Then the policy is infeasible in May and June. In this case the constraint means that the slope of the cumulative production curve is bounded by 1,800. One solution in this case would be to produce 980 in each of the first four months and 1,800 units in each of the last two months. Another constraint might be that the maximum change from one month to the next be no more than 750 units. Suggest a production plan to meet these constraints.

These are a few examples of constraints that might arise in using aggregate planning methodology. As the constraints become more complex, finding good solutions graphically becomes more difficult. Fortunately, most constraints of this nature can be incorporated easily into the linear programming formulations of aggregate planning problems.

Problems for Section 3.4

9. Harold Grey owns a small farm in the Salinas Valley that grows apricots. The apricots are dried on the premises and sold to a number of large supermarket chains. Based on past experience and committed contracts, he estimates that sales over the next five years in thousands of packages will be as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>Forecasted Demand (thousands of packages)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>180</td>
</tr>
<tr>
<td>2</td>
<td>120</td>
</tr>
<tr>
<td>3</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
</tr>
<tr>
<td>5</td>
<td>170</td>
</tr>
</tbody>
</table>

Assume that each worker stays on the job for at least one year, and that Grey currently has three workers on the payroll. He estimates that he will have 20,000 packages on hand at the end of the current year. Assume that, on the average, each worker is paid $25,000 per year and is responsible for producing 30,000 packages. Inventory costs have been estimated to be 4 cents per package per year, and shortages are not allowed.

Based on the effort of interviewing and training new workers, Farmer Grey estimates that it costs $500 for each worker hired. Severance pay amounts to $1,000 per worker.

a. Assuming that shortages are not allowed, determine the minimum constant workforce that will be needed over the next five years.

b. Evaluate the cost of the plan found in part (a).

10. For the data given in Problem 9, graph the cumulative net demand.

a. Graphically determine a production plan that changes the production rate exactly once during the five years, and evaluate the cost of that plan.

b. Graphically determine a production plan that changes the production rate exactly twice during the five years, and evaluate the cost of that plan.

11. An implicit assumption made in Problem 9 was that dried apricots unsold at the end of a year could be sold in subsequent years. Suppose that apricots unsold at the end of any year must be discarded. Assume a disposal cost of $0.20 per package. Resolve Problem 9 under these conditions.

12. The personnel department of the A&M Corporation wants to know how many workers will be needed each month for the next six-month production period. The following is a monthly demand forecast for the six-month period.

<table>
<thead>
<tr>
<th>Month</th>
<th>Forecasted Demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>1,250</td>
</tr>
<tr>
<td>August</td>
<td>1,100</td>
</tr>
<tr>
<td>September</td>
<td>950</td>
</tr>
<tr>
<td>October</td>
<td>900</td>
</tr>
<tr>
<td>November</td>
<td>1,000</td>
</tr>
<tr>
<td>December</td>
<td>1,150</td>
</tr>
</tbody>
</table>
Chapter Three  Aggregate Planning

The inventory on hand at the end of June was 500 units. The company wants to maintain a minimum inventory of 300 units each month and would like to have 400 units on hand at the end of December. Each unit requires five employee-hours to produce, there are 20 working days each month, and each employee works an eight-hour day. The workforce at the end of June was 35 workers.

a. Determine a minimum inventory production plan (i.e., one that allows arbitrary hiring and firing).

b. Determine the production plan that meets demand but does not hire or fire workers during the six-month period.

13. Mr. Meadows Cookie Company makes a variety of chocolate chip cookies in its plant in Albion, Michigan. Based on orders received and forecasts of buying habits, it is estimated that the demand for the next four months is 850, 1,260, 510, and 980, expressed in thousands of cookies. During a 46-day period when there were 120 workers, the company produced 1.7 million cookies. Assume that the number of workdays over the four months are respectively 26, 24, 20, and 16. There are 100 workers employed, and there is no starting inventory of cookies.

a. What is the minimum constant workforce required to meet demand over the next four months?

b. Assume that $c_P = 10$ cents per cookie per month, $c_R = $100, and $c_F = $200.

Evaluate the cost of the plan derived in part (a).

14. A local semiconductor firm, Superchip, is planning its workforce and production levels over the next year. The firm makes a variety of microprocessors and uses sales dollars as its aggregate production measure. Based on orders received and sales forecasts provided by the marketing department, the estimate of dollar sales for the next year by month is as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Production Days</th>
<th>Predicted Demand (in $10,000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>January</td>
<td>22</td>
<td>340</td>
</tr>
<tr>
<td>February</td>
<td>16</td>
<td>380</td>
</tr>
<tr>
<td>March</td>
<td>21</td>
<td>220</td>
</tr>
<tr>
<td>April</td>
<td>19</td>
<td>100</td>
</tr>
<tr>
<td>May</td>
<td>23</td>
<td>490</td>
</tr>
<tr>
<td>June</td>
<td>20</td>
<td>620</td>
</tr>
<tr>
<td>July</td>
<td>24</td>
<td>375</td>
</tr>
<tr>
<td>August</td>
<td>12</td>
<td>310</td>
</tr>
<tr>
<td>September</td>
<td>19</td>
<td>175</td>
</tr>
<tr>
<td>October</td>
<td>22</td>
<td>145</td>
</tr>
<tr>
<td>November</td>
<td>20</td>
<td>120</td>
</tr>
<tr>
<td>December</td>
<td>16</td>
<td>150</td>
</tr>
</tbody>
</table>

Inventory holding costs are based on a 25 percent annual interest charge. It is anticipated that there will be 675 workers on the payroll at the end of the current year and inventories will amount to $120,000. The firm would like to have at least $100,000 of inventory at the end of December next year. It is estimated that each worker accounts for an average of $60,000 of production per year (assume that one year consists of 250 working days). The cost of hiring a new worker is $200, and the cost of laying off a worker is $400.

3.5 SOLUTION OF AGGREGATE PLANNING PROBLEMS BY LINEAR PROGRAMMING

Linear programming is a term used to describe a general class of optimization problems. The objective is to determine values of nonnegative real variables in order to maximize or minimize a linear function of these variables that is subject to linear constraints of these variables. The primary advantage in formulating a problem as a linear program is that optimal solutions can be found very efficiently by the simplex method. When all cost functions are linear, there is a linear programming formulation of the general aggregate planning problem. Because of the efficiency of commercial linear programming codes, this means that (essentially) optimal solutions can be obtained for very large problems.

Cost Parameters and Given Information

The following values are assumed to be known:

- $c_h = $30,000/year (cost of hiring one worker)
- $c_f = $10,000/year (cost of firing one worker)
- $c_t = $100 (cost of holding one unit of stock for one period)
- $c_p = $120 (cost of producing one unit on regular time)
- $c_o = $20,000/unit (incremental cost of producing one unit on overtime)
- $c_i = $200 (idle cost per unit of production)
- $c_s = $30,000/unit (cost to subcontract one unit of production)
- $n_p = 100 (number of production days per period)
- $n_r = 100 (number of aggregate units produced by one worker in one day)
- $I_0 = 100,000 (initial inventory on hand at the start of the planning horizon)
- $P_0 = 500,000 (initial workforce at the start of the planning horizon)
- $D = 120,000 (forecast of demand in period 1)

The cost parameters also may be time-dependent; that is, they may change with $t$. Time-dependent cost parameters could be useful for modeling changes in the costs of hiring or firing due, for example, to shortages in the labor pool, or changes in the costs of production and/or storage due to shortages in the supply of resources, or changes in interest rates.

1 An overview of linear programming can be found in Supplement 2, which follows this chapter.
2 The qualifier is included because rounding may give suboptimal solutions. There will be more about this point later.
Problem Variables

The following are the problem variables:

- \( W_t \) = Workforce level in period \( t \)
- \( P_t \) = Production level in period \( t \)
- \( I_t \) = Inventory level in period \( t \)
- \( H_t \) = Number of workers hired in period \( t \)
- \( F_t \) = Number of workers fired in period \( t \)
- \( O_t \) = Overtime production in units, \( U_t \) = Worker idle time in units ("undertime")
- \( S_t \) = Number of units subcontracted from outside.

The overtime and idle time variables are determined in the following way. The term \( K_nW_t \) represents the number of units produced by one worker in period \( t \), so that \( K_nW_t \) would be the number of units produced by the entire workforce in period \( t \). However, we do not require that \( K_nW_t = P_t \). If \( P_t > K_nW_t \), then the number of units produced exceeds what the workforce can produce on regular time. This means that the difference is being produced on overtime, so that the number of units produced on overtime is exactly \( O_t = P_t - K_nW_t \). If \( P_t < K_nW_t \), then the workforce is producing less than it should be on regular time, which means that there is worker idle time. The idle time is measured in units of production rather than in time, and is given by \( U_t = K_nW_t - P_t \).

Problem Constraints

Three sets of constraints are required for the linear programming formulation. They are included to ensure that conservation of labor and conservation of units are satisfied.


\[
W_t = \text{Number of workers hired in period } t - \text{Number of workers fired in period } t + W_{t-1} + H_t - F_t \quad \text{for } 1 \leq t \leq T.
\]

2. Conservation of units constraints.

\[
I_t = \text{Inventory level in period } t - \text{Demand in period } t + I_{t-1} + P_t - D_t \quad \text{for } 1 \leq t \leq T.
\]

3. Constraints relating production levels to workforce levels.

\[
P_t = \text{Number of units produced by regular workforce in period } t + \text{Number of units produced on overtime in period } t + U_t + S_t \quad \text{for } 1 \leq t \leq T.
\]

In addition to these constraints, linear programming requires that all problem variables be nonnegative. These constraints and the nonnegativity constraints are the minimum that must be present in any formulation. Notice that (1), (2), and (3) constitute 3\( T \) constraints, rather than 3 constraints, where \( T \) is the length of the forecast horizon.

The formulation also requires specification of the initial inventory, \( I_0 \), and the initial workforce, \( W_0 \), and may include specification of the ending inventory in the final period, \( I_T \).

The objective function includes all the costs defined earlier. The linear programming formulation is to choose values of the problem variables \( W_t, P_t, I_t, H_t, F_t, O_t, U_t, S_t \) to

\[
\text{Minimize } \sum_{t=1}^{T} (c_H H_t + c_F F_t + c_I I_t + c_O O_t + c_U U_t + c_S S_t)
\]

subject to

\[
W_t = W_{t-1} + H_t - F_t \quad \text{for } 1 \leq t \leq T \quad \text{(conservation of workforce)},
\]

\[
P_t = K_nW_t + O_t - U_t \quad \text{for } 1 \leq t \leq T \quad \text{(production and workforce)},
\]

\[
I_t = I_{t-1} + P_t + S_t - D_t \quad \text{for } 1 \leq t \leq T \quad \text{(inventory balance)},
\]

\[
H_t, P_t, I_t, O_t, U_t, S_t, W_t, P_t \geq 0 \quad \text{(nonnegativity)}.
\]

plus any additional constraints that define the values of starting inventory, starting workforce, ending inventory, or any other variables with values that are fixed in advance.

Rounding the Variables

In general, the optimal values of the problem variables will not be integers. However, fractional values for many of the variables do not make sense. These variables include the size of the workforce, the number of workers hired each period, and the number of workers fired each period, and also may include the number of units produced each period. It is possible that fractional numbers of units could be produced in some applications. One way to deal with this problem is to require in advance that some or all of the problem variables assume only integer values. Unfortunately, this makes the solution algorithm considerably more complex. The resulting problem, known as an integer linear programming problem, requires much more computational effort to solve than does ordinary linear programming. For a moderate-sized problem, solving the problem as an integer linear program is certainly a reasonable alternative.

If an integer programming code is unavailable or if the problem is simply too large to solve by integer programming, linear programming still provides a workable solution. However, after the linear programming solution is obtained, some of the problem variables must be rounded to integer values. Simply rounding off each variable to the closest integer may lead to an infeasible solution and/or in one in which production and workforce levels are inconsistent. It is not obvious what is the best way to round the variables. We recommend the following conservative approach: round the values of the numbers of workers in each period to the next larger integer. Once the values of \( W_t \) are determined, the values of the other variables, \( H_t, P_t, I_t \), can be found along with the cost of the resulting plan.

Conservative rounding will always result in a feasible solution, but will rarely give the optimal solution. The conservative solution generally can be improved by trial-and-error experimentation.
There is no guarantee that if a problem can be formulated as a linear program, the final solution makes sense in the context of the problem. In the aggregate planning problem, it does not make sense that there should be both overtime production and idle time in the same period. This means that either one or both of the variables $O_t$ and $F_t$ must be zero, and either one or both of the variables $H_t$ and $F_t$ must be zero for each $t$, $1 \leq t \leq T$. This requirement can be included explicitly in the problem formulation by adding the constraints

$$O_t U_t = 0 \quad \text{for } 1 \leq t \leq T,$$
$$H_t F_t = 0 \quad \text{for } 1 \leq t \leq T,$$

since if the product of two variables is zero it means that at least one must be zero. Unfortunately, these constraints are not linear, as they involve a product of problem variables. However, it turns out that it is not necessary to explicitly include these constraints, because the optimal solution to a linear programming problem always occurs at an extreme point of the feasible region. It can be shown that every extreme point solution automatically has this property. If this were not the case, the linear programming solution would be meaningless.

**Extensions**

Linear programming also can be used to solve somewhat more general versions of the aggregate planning problem. Uncertainty of demand can be accounted for indirectly by assuming that there is a minimum buffer inventory $B_t$ each period. In that case we would include the constraints

$$I_t \geq B_t \quad \text{for } 1 \leq t \leq T.$$

The constants $B_t$ would have to be specified in advance. Upper bounds on the number of workers hired and the number of workers fired each period could be included in a similar way. Capacity constraints on the amount of production each period could easily be represented by the set of constraints:

$$P_t \leq C_t \quad \text{for } 1 \leq t \leq T.$$

The linear programming formulation introduced in this section assumed that inventory levels would never go negative. However, in some cases it might be desirable or even necessary to allow demand to exceed supply, for example, if forecast demand exceeded production capacity over some set of planning periods. In order to treat backlogging of excess demand, the inventory level $I_t$ must be expressed as the difference between two nonnegative variables, say $I_t^+$ and $I_t^-$, satisfying

$$I_t = I_t^+ - I_t^-,$$
$$I_t^+ \geq 0, \quad I_t^- \geq 0.$$

The holding cost would now be charged against $I_t^+$ and the penalty cost for backorders (say $c_p$) against $I_t^-$. However, notice that for the solution to be sensible, it must be true that $I_t^+$ and $I_t^-$ are not both positive in the same period $t$. As with the overtime and idle time and the hiring and firing variables, the properties of linear programming will guarantee that this holds without having to explicitly include the constraint $I_t^+ I_t^- = 0$ in the formulation.

**FIGURE 3-5**

A convex piecewise-linear function

In the development of the linear programming model, we stated the requirement that all the cost functions must be linear. This is not strictly correct. Linear programming also can be used when the cost functions are convex piecewise-linear functions.

A convex function is one with an increasing slope. A piecewise-linear function is one that is composed of straight-line segments. Hence, a convex piecewise-linear function is a function composed of straight lines that have increasing slopes. A typical example is presented in Figure 3-5.

In practice, it is likely that some or all of the cost functions for aggregate planning are convex. For example, if Figure 3-5 represents the cost of hiring workers, then the marginal cost of hiring one additional worker increases with the number of workers that have already been hired. This is probably more accurate than assuming that the cost of hiring one additional worker is a constant independent of the number of workers previously hired. As more workers are hired, the available labor pool shrinks and more effort must be expended to hire the remaining available workers.

In order to see exactly how convex piecewise-linear functions would be incorporated into the linear programming formulation, we will consider a very simple case. Suppose that the cost of hiring new workers is represented by the function pictured in Figure 3-6. According to the figure, it costs $c_{H1}$ to hire each worker until $H^*$ workers are hired, and it costs $c_{H2}$ for each worker hired beyond $H^*$ workers, with $c_{H1} < c_{H2}$. The variable $H_t$, the number of workers hired in period $t$, must be expressed as the sum of two variables:

$$H_t = H_{t1} + H_{t2}.$$

Interpret $H_{t1}$ as the number of workers hired up to $H^*$ and $H_{t2}$ as the number of workers hired beyond $H^*$ in period $t$. The cost of hiring is now represented in the objective function as

$$\sum_{t=1}^{T} (c_{H1} H_{t1} + c_{H2} H_{t2}).$$
and the additional constraints

\[ H_t = H_{1t} + H_{2t} \]
\[ 0 \leq H_{1t} \leq H^* \]
\[ 0 \leq H_{2t} \]

must also be included.

In order for the final solution to make sense, it can never be the case that \( H_{1t} < H^* \) and \( H_{2t} > 0 \) for some \( t \). (Why?) However, because linear programming searches for the minimum cost solution, it will force \( H_{1t} \) to its maximum value before allowing \( H_{2t} \) to become positive, since \( c_{1t} \leq c_{2t} \). This is the reason that the cost functions must be convex. This approach can easily be extended to more than two linear segments and to any of the other cost functions present in the objective function. The technique is known as separable convex programming and is discussed in greater detail in Hillier and Lieberman (1990).

Other Solution Methods

Bowman (1956) suggested a transportation formulation of the aggregate planning problem when back orders are not permitted. Several authors have explored solving aggregate planning problems with specially tailored algorithms. These algorithms could be faster for solving large aggregate planning problems than a general linear programming code such as LINDO. In addition, some of these formulations allow for certain types of nonlinear costs. For example, if there are economies of scale in production, the production cost function is likely to be a concave function of the number of units produced. Concave cost functions are not as amenable to linear programming formulations as convex functions. We believe that a general-purpose linear programming package is sufficient for most reasonable-sized problems that one is likely to encounter in the real world. However, the reader should be aware that there are alternative solution techniques that could be more efficient for large problems and could provide solutions when some of the cost functions are nonlinear. A paper that details a transportation-type procedure more efficient than the simplex method for solving aggregate planning problems is Erenguc and Tufekci (1988).

3.6 SOLVING AGGREGATE PLANNING PROBLEMS BY LINEAR PROGRAMMING: AN EXAMPLE

We will demonstrate the use of linear programming by finding the optimal solution to the example presented in Section 3.4. As there is no subcontracting, overtime, or idle time allowed, and the cost coefficients are constant with respect to time, the objective function is simply

\[
\text{Minimize} \left( 500 \sum_{t=1}^{6} H_t + 1000 \sum_{t=1}^{6} P_t + 80 \sum_{t=1}^{6} I_t \right)
\]

The boundary conditions comprise the specifications of the initial inventory of 500 units, the initial workforce of 300 workers, and the ending inventory of 600 units. These are best handled by including a separate additional constraint for each boundary condition.

The constraints are obtained by substituting \( t = 1, \ldots, 6 \) into Equations (A), (B), and (C). The full set of constraints expressed in standard linear programming format (with all problem variables on the left-hand side and nonnegative constants on the right-hand side) is as follows:

\[
W_1 - W_0 - H_1 + P_1 = 0, \\
W_2 - W_1 - H_2 + P_2 = 0, \\
W_3 - W_2 - H_3 + P_3 = 0, \\
W_4 - W_3 - H_4 + P_4 = 0, \\
W_5 - W_4 - H_5 + P_5 = 0, \\
W_6 - W_5 - H_6 + P_6 = 0; \tag{A}
\]

\[
P_1 - I_1 + I_0 = 1,280, \\
P_2 - I_2 + I_1 = 640, \\
P_3 - I_3 + I_2 = 900, \\
P_4 - I_4 + I_3 = 1,200, \\
P_5 - I_5 + I_4 = 2,000, \\
P_6 - I_6 + I_5 = 1,400; \tag{B}
\]

\[
P_1 - 2.931W_1 = 0, \\
P_2 - 3.517W_1 = 0, \\
P_3 - 2.638W_1 = 0, \\
P_4 - 3.810W_1 = 0, \\
P_5 - 3.224W_1 = 0, \\
P_6 - 2.198W_1 = 0; \tag{C}
\]
We have solved this linear program using the LINDO system developed by Schrage (1984). The output of the LINDO program is given in Table 3-5. In the supplement on linear programming, we also treat solving linear programs with Excel. The Excel Solver, of course, gives the same results. The value of the objective function at the optimal solution is $379,320.90, which is considerably less than that achieved with either the zero inventory plan or the constant workforce plan. However, this cost is based on fractional values of the variables. The actual cost will be slightly higher after rounding.

Following the rounding procedure recommended earlier, we will round all the values of $W_i$ to the next higher integer. That gives $W_1 = \ldots = W_4 = 273$ and $W_5 = W_6 = 738$. This determines the values of the other problem variables. This means that the firm should fire 27 workers in January and hire 465 workers in May. The complete solution is given in Table 3-6.

Again, because column H in Table 3-6 corresponds to net demand, we add the 600 units of ending inventory in June, giving a total inventory of 900 + 600 = 1,500 units. Hence, the total cost of this plan is $(500)(465) + (1000)(27) + (80)(1500) = 379,500$, which represents a substantial savings over both the zero inventory plan and the constant workforce plan.

The results of the linear programming analysis suggest another plan that might be more suitable for the company. Because the optimal strategy is to decrease the workforce in January and build it back up again in May, a reasonable alternative might be to hire a 27 workers in January and to hire fewer workers in May. In this case, the most efficient method for finding the correct number of workers to hire in May is to simply re-solve the linear program, but without the variables $F_1, \ldots, F_6$, as no firing of workers means that these variables are forced to zero. (If you wish to avoid reentering the problem into the computer, simply add the old formulation with the constraints $F_1 = 0, F_2 = 0, \ldots, F_6 = 0$.) The optimal number of workers to hire in May turns out to be 374 if no workers are fired, and the cost of the plan is approximately $386,120. This is only slightly more expensive than the optimal plan, and has the important advantage of not requiring the firing of any workers.

**Problems for Sections 3.5 and 3.6**

17. Formulate Problem 13 as a linear program. Be sure to define all variables and include all the required constraints.
Chapter Three  Aggregate Planning

18. Problem 17 was solved on the computer. The linear programming output of the program is as follows:

**LP OPTIMUM FOUND AT STEP 14**

<table>
<thead>
<tr>
<th>OBJECTIVE FUNCTION VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) 36185.0600</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>VARIABLE</th>
<th>VALUE</th>
<th>REDUCED COST</th>
</tr>
</thead>
<tbody>
<tr>
<td>H1 6.143851</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>H2 64.310690</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>H3 0.000000</td>
<td>300.000000</td>
<td></td>
</tr>
<tr>
<td>H4 116.071400</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>F1 0.000000</td>
<td>300.000000</td>
<td></td>
</tr>
<tr>
<td>F2 0.000000</td>
<td>300.000000</td>
<td></td>
</tr>
<tr>
<td>F3 87.662330</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>F4 0.000000</td>
<td>300.000000</td>
<td></td>
</tr>
<tr>
<td>I1 0.000000</td>
<td>59.415580</td>
<td></td>
</tr>
<tr>
<td>I2 0.000000</td>
<td>189.285700</td>
<td></td>
</tr>
<tr>
<td>E3 0.000000</td>
<td>31.006490</td>
<td></td>
</tr>
<tr>
<td>W1 106.143900</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>W2 170.454500</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>W3 82.792210</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>W4 198.863600</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>P1 850.000000</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>P2 1260.000000</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>P3 510.000000</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>P4 980.000000</td>
<td>.000000</td>
<td></td>
</tr>
<tr>
<td>I4 0.000000</td>
<td>120.292200</td>
<td></td>
</tr>
</tbody>
</table>

19. a. Formulate Problem 9 as a linear program.
   b. Solve the problem using a linear programming code. Round the variables in the resulting solution and determine the cost of the plan you obtain.

20. a. Formulate Problem 14 as a linear program.
   b. Solve the problem using a linear programming code. Round off the solution and determine the cost of the resulting plan.

21. Consider Mr. Meadows Cookie Company, described in Problem 13. Suppose that the cost of hiring workers each period is $100 for each worker until 20 workers are hired, $400 for each worker when between 21 and 50 workers are hired, and $700 for each worker hired beyond 50.

22. a. Write down the complete linear programming formulation of the revised problem.
   b. Solve the revised problem using a linear programming code. What difference does the new hiring cost function make in the solution?

23. a. Formulate the problem of optimizing Leather-All's hiring schedule as a linear program. Define all problem variables and include whatever constraints are necessary.
   b. Solve the problem formulated in part (a) using a linear programming code. Round all the relevant variables and determine the cost of the resulting plan. Compare your results with those obtained in Problem 22(c) and (d).
3.7 THE LINEAR DECISION RULE

An interesting alternative approach to solving the aggregate planning problem has been suggested by Holt, Modigliani, Muth, and Simon (1960). They assume that all the relevant costs, including inventory costs and the costs of changing production levels and numbers of workers, are represented by quadratic functions. That is, the total cost over the T-period planning horizon can be written in the form

$$\sum_{t=1}^{T} \left[ c_1 W_t + c_2 (W_t - W_{t-1})^2 + c_3 (P_t - D_t)^2 + c_4 P_t + c_5 (L_t - a_0)^2 \right]$$

subject to

$$L_t = L_{t-1} + P_t - D_t \quad \text{for } 1 \leq t \leq T.$$  

The values of the constants $$c_1, c_2, \ldots, c_6$$ must be determined for each particular application. Expressing the cost functions as quadratic functions rather than simple linear functions has some advantages over the linear programming approach. As quadratic functions are differentiable, the standard rules of calculus can be used to determine optimal solutions. The optimal solution will occur where the first partial derivatives with respect to each of the problem variables are zero. When quadratic functions are differentiated, they yield first-order equations (linear equations), which are easy to solve. Also, quadratic functions give a more accurate approximation to general nonlinear functions than do linear functions. (Any nonlinear function may be approximated by a Taylor series expansion in which the first two terms yield a quadratic approximation.)

However, the quadratic approach also has one serious disadvantage. It is that quadratic functions (that is, parabolas) are symmetric (see Figure 3-7). Hence, the cost of hiring a given number of workers must be the same as the cost of firing the same number of workers, and the cost of producing a given number of units on overtime must be the same as the cost attached to the same number of units of worker idle time. The problem can be overcome somewhat by not setting the center of symmetry of the cost function at zero, but the basic problem of symmetry still prevails.

The most appealing feature of this approach is the simple form of the optimal policy. For example, the optimal production level in period $$t$$, $$P_t$$, has the form

$$P_t = \sum_{k=0}^{K} (a_k D_{t+k} + b W_{t+k} + c_t W_t + d)$$

The terms $$a_k, b, c,$$ and $$d$$ are constants that depend on the cost parameters. $$W_t$$ has a similar form. The computational advantages of the linear decision rule are less important now than they were when this analysis was first done because of the widespread availability of computers today. Because linear programming provides a much more flexible framework for formulating aggregate planning problems, and reasonably large linear programs can be solved even on personal computers, one would have to conclude that linear programming is a preferable solution technique.

It should be recognized, however, that the text by Holt, Modigliani, Muth, and Simon represents a landmark work in the application of quantitative methods to production planning problems. The authors developed a solution method that results in a set of formulas that are easy to implement, and they actually undertook the implementation of the method. The work details the application of the approach to a large manufacturer of household paints in the Pittsburgh area. The analysis was actually implemented in the company, but a subsequent visit to the firm indicated that serious problems arose when the linear decision rule was followed, primarily because of the firm's policy of not firing workers when the model indicated that they should be fired. (See Vollmann, Berry, and Whybark, 1992, p. 627.)

3.8 MODELING MANAGEMENT BEHAVIOR

Bowman (1963) developed an interesting technique for aggregate planning that should be applicable to other types of problems as well. His idea is to construct a sensible model for controlling production levels and fit the parameters of the model as closely as possible to the actual prior decisions made by management. In this way the model reflects the judgment and the experience of managers and avoids a number of the problems that arise when using more traditional modeling methods. One such problem is determining the accuracy of the assumptions required by the model. Another problem that is avoided is the need to determine values of input parameters that could be difficult to measure.

Let us consider the problem of producing a single product over T planning periods. Suppose that $$D_1, \ldots, D_T$$ are the forecasts of demand for the next T periods and that production levels $$P_1, \ldots, P_T$$ must be determined. The simplest reasonable decision rule is

$$P_t = D_t \quad \text{for } 1 \leq t \leq T.$$  

Trying to match production exactly to demand will generally result in a production schedule that is too erratic. Production smoothing can be accomplished using a decision rule of the form

$$P_t = D_t + \alpha (P_{t-1} - D_t),$$
where \( \alpha \) is a smoothing factor for production, having the property \( 0 \leq \alpha \leq 1 \). (The effect of smoothing here is the same as that of exponential smoothing, discussed in detail in Chapter 2.) When \( \alpha \) equals zero, this rule is identical to the one just presented. When \( \alpha = 1 \), the production in period \( t \) is exactly the same as the production in period \( t-1 \). The choice of \( \alpha \) provides a means of placing a relative weight on matching production with demand versus holding production constant from period to period.

In addition to smoothing production, the firm also might be interested in keeping inventory levels near some target level, say \( I_0 \). A model that smooths both inventory and production simultaneously is

\[
P_t = D_t + \alpha(P_{t-1} - D_t) + \beta(I_t - I_{t-1}),
\]

where \( 0 \leq \beta \leq 1 \) measures the relative weight placed on smoothing of inventory.

Finally, the model should incorporate demand forecasts. In this way, production levels can be increased or decreased in anticipation of a change in the pattern of demands. A rule that includes smoothing of production and inventory as well as forecasts of future demand is

\[
P_t = \sum_{k=1}^{\infty} a_k D_{t+k} + \alpha(P_{t-1} - D_t) + \beta(I_t - I_{t-1}).
\]

This decision rule is arrived at by a straightforward common-sense analysis of what a good production rule should be. It is interesting to note how similar this rule is to the linear decision rule discussed in Section 3.7, which was derived from a mathematical model. Undoubtedly, the previous work on the linear decision rule inspired the form of this rule. It is often the case that the most valuable feature of a model is indicating the best form of an optimal strategy, and not necessarily the best strategy itself.

This particular model for \( P_t \) requires determination of \( a_1, \ldots, a_n, \alpha, \beta, \) and \( I_0 \). Bowman suggests that the values of these parameters be determined by retrospectively observing the system for a reasonable period of time, and fitting the parameters to the actual history of management actions during that time using a technique such as least squares. In this way the model becomes a reflection of past management behavior.

Bowman compares the actual experience of a number of companies with the experience that they would have had using his approach, and shows that in most cases there would have been a substantial cost reduction. The model merely emulates management behavior, so why should it outperform actual management experience? The theory is that a model derived in this manner is a reflection of management making rational decisions, as most of the time the system is stable. However, if a sudden unusual event occurs, such as a much-higher-than-anticipated demand or a sudden decline in productive capacity due, for example, to the breakdown of a machine or the loss of key personnel, it is likely that many managers would tend to overreact. However, the model would recommend production levels that are consistent with past decision making. Hence, the use of a simple but consistent model for making decisions would keep management from panicking in an unusual situation. Although the approach is conceptually appealing, there are few reports in the literature of successful implementations of such a technique.

### 3.9 DISAGGREGATING AGGREGATE PLANS

As we saw earlier in this chapter, aggregate planning may be done at several levels of the firm. Example 3.1 considered a single plant, and showed how one might define an aggregate unit to include the six different items produced at that plant. Aggregate

Problems for Sections 3.7 and 3.8

24. The form of the optimal production rule derived by Holt, Modigliani, Math, and Simon (1960) for the case of the paint company indicates that the production level in period \( t \) should be computed from the formula

\[
P_t = 0.463D_t + 0.234D_{t-1} + 0.111D_{t-2} + 0.046D_{t-3} + 0.013D_{t-4} - 0.22D_{t-5} - 0.088D_{t-6} - 0.01D_{t-7} - 0.009D_{t-8} - 0.008D_{t-9} - 0.007D_{t-10} + 0.005D_{t-11} + 0.993W_{t-1} + 0.464I_{t-1} + 153,
\]

and the workforce level in period \( t \) from the formula

\[
W_t = 0.101D_t + 0.0882D_{t-1} + 0.071D_{t-2} + 0.054D_{t-3} + 0.042D_{t-4} + 0.031D_{t-5} + 0.023D_{t-6} + 0.016D_{t-7} + 0.012D_{t-8} + 0.009D_{t-9} + 0.006D_{t-10} + 0.005D_{t-11} + 0.743W_{t-1} - 0.01I_{t-1} - 2.09.
\]

Suppose that forecast demands for the coming 12 months are 150, 164, 185, 193, 167, 143, 126, 93, 84, 72, 50, 66. The current number of employees is 180, and the current inventory level is 45 aggregate units.

a. Compute the values of the aggregate production level and the number of workers that the company should be using in the current period.

b. Repeat the calculation you performed in part (a) but ignore all terms with a factor that has an absolute value less than or equal to .01 when calculating \( P_t \) and less than or equal to .005 when calculating \( W_t \). How much difference do you observe in the answer?

c. Suppose that the current demand is realized to be 163 and the new forecast one year ahead is 72. Using the approximation suggested in part (b), determine the new values for the aggregate production and the number of workers. [Hint: You must compute the new value \( I_{n+1} \) and assume that the new value of \( W_{t-1} \) is the value computed in part (a).]

25. Using the following values of management coefficients for Bowman's smoothed production model, determine the production level for which Harold Grey (described in Problem 9) should plan in the coming year. Assume that the current production level is 150,000 packages.

\[
a_1 = 3475, \quad a_2 = 1211, \quad a_3 = 0.556, \quad a_4 = 0.663, \quad a_5 = 0.023, \quad \alpha = 0.6, \quad \beta = 3, \quad I_0 = 40,000.
\]

26. a. Discuss the similarities and the differences between the linear decision rule of Holt, Modigliani, Math, and Simon and Bowman's management coefficients approach.

b. Discuss the relative advantages and disadvantages of linear decision rules versus linear programming for solving aggregate planning problems.
planning might be done for a single plant, a product family, a group of families, or for the firm as a whole.

There are two views of production planning: bottom-up or top-down. The bottom-up approach means that one would start with individual item production plans. These plans could then be aggregated up the chain of products to produce aggregate plans. The top-down approach, which is the one treated in this chapter, is to start with an aggregate plan at a high level. These plans would then have to be "disaggregated" to produce detailed production plans at the plant and individual item levels.

It is not clear that disaggregation is an issue in all circumstances. If the aggregate plan is used only for macro planning purposes, and not for planning at the detail level, then one need not worry about disaggregation. However, if it is important that individual item production plans and aggregate plans be consistent, then it might be necessary to consider disaggregation schemes.

The disaggregation problem is similar to the classic problem of resource allocation. Consider how resources are allocated in a university, for example. A university receives revenues from tuition, gifts, interest on the endowment, and research grants. Costs include salaries, maintenance, and capital expenditures. Once an annual budget is determined, each school (arts and science, engineering, business, law, etc.) and each budget center (staff, maintenance, buildings and grounds, etc.) would have to be allocated its share. Budget centers would have to allocate funds to each of their subgroups. For example, each school would allocate funds to individual departments, and departments would allocate funds to faculty and staff in that department.

In the manufacturing context, a disaggregation scheme is just a means of allocating aggregate units to individual items. Just as funds are allocated on several levels in the university, aggregate units might have to be disaggregated at several levels of the firm. This is the idea behind hierarchical production planning championed by several researchers at MIT and reported in detail in Bitran and Hax (1981) and Hax and Candea (1984).

We will discuss one possible approach to the disaggregation problem from Bitran and Hax (1981). Suppose that \( X \) represents the number of aggregate units of production for a particular planning period. Further, suppose that \( X \) represents an aggregation of \( n \) different items \( Y_1, Y_2, \ldots, Y_n \). The question is how to divide up (i.e., disaggregate) \( X \) among the \( n \) items. We know that holding costs are already included in the determination of \( X \), so we need not include them again in the disaggregation scheme. Suppose that \( K_j \) represents the fixed cost of setting up production of \( Y_j \), and \( A_j \) is the annual usage rate for item \( j \). A reasonable optimization criterion in this context is to choose \( Y_1, Y_2, \ldots, Y_n \) to minimize the average annual cost of setting up for production. As we will see in Chapter 4, the average annual setup cost for item \( j \) is \( K_j A_j Y_j \). Hence, disaggregation requires solving the following mathematical programming problem:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{j=1}^{n} K_j A_j Y_j \\
\text{subject to} & \quad \sum_{j=1}^{n} Y_j = X \\
\text{and} & \quad a_j \leq Y_j \leq b_j \quad \text{for} \quad 1 \leq j \leq n.
\end{align*}
\]

The upper and lower bounds on \( Y_j \) account for possible side constraints on the production level for item \( j \).

A number of feasibility issues need to be addressed before the family run sizes \( Y_j \) are further disaggregated into lots for individual items. The objective is to schedule the lots for individual items within a family so that they run out at the scheduled setup time for the family. In this way, items within the same family can be produced within the same production setup.

The concept of disaggregating the aggregate plan along organizational lines in a fashion that is consistent with the aggregation scheme is an appealing one. Whether or not the methods discussed in this section provide a workable link between aggregate plans and detailed item schedules remains to be seen.

Another approach to the disaggregation problem has been explored by Chung and Krajewski (1984). They develop a mathematical programming formulation of the problem. Inputs to the program include aggregate plans for each product family. This includes setup time, setup status, total production level for the family, inventory level, workforce level, overtime, and regular time availability. The goal of the analysis is to specify lot sizes and timing of production runs for each individual item, consistent with the aggregate information for the product family. Although such a formulation

### Snapshot Application

**WELCH'S USES AGGREGATE PLANNING FOR PRODUCTION SCHEDULING**

Welch's is a make-to-stock food manufacturer based in Concord, Massachusetts. They are probably best known for grape jelly and grape juices, but they produce a wide variety of processed foods. Allen and Schuster (1990) describe an aggregate planning model for Welch's primary production facility. The characteristics of the production system for which their system was designed are:

- Dynamic, uncertain demand, resulting in changing buffer stock requirements.
- Make-to-stock environment.
- Dedicated production lines.
- Production lines that each produces two or more families of products.
- Large setup times and setup costs for families, as opposed to low setup times and costs for individual items.

The two primary objectives of the production system, as described by the authors, are to smooth peak demands through time so as not to exceed production capacity and to allocate production requirements among the families to balance family holding and setup costs. The planning is done with a six-month time horizon for demand forecasting. The six-month period is divided into two portions: the next four months and the remaining five months. Detailed plans are developed for the near term, including regular and overtime production allocation.

The model has two primary components: a family planning model, which finds the optimal timing and sizing of family production runs, and a disaggregation planning model, which takes the results of family planning and determines lot sizes for individual items within families.

The authors also discuss several implementation issues, specific to Welch's environment. Product run lengths must be tied to the existing eight-hour shift structure. To do so, they recommend that production run lengths be expressed as multiples of one-quarter shift hours.

The model was implemented on a personal computer. Computing times are very moderate. Solution techniques include a mixed integer mathematical program and a linear programming formulation with relaxation (that is, rounding of variables to integer values).

This case demonstrates that the concept discussed in this chapter can be useful in a real production planning environment. Although the system described here is not based on any of the specific models discussed in this chapter, this application shows that aggregation and disaggregation are useful concepts. Hierarchical aggregation for production scheduling is a valuable planning tool.
3.1 Ο PRODUCTIΟΝ PLANNING

Section 3.9

Problems for Section 3.9

27. What does "disaggregation of aggregate plans" mean?
28. Discuss the following quotation made by a production manager: "Aggregate planning is useless to me because the results have nothing to do with my master production schedule."

3.10 PRODUCTION PLANNING ON A GLOBAL SCALE

Globalization of manufacturing operations is commonplace. Many major corporations are now classified as multinationals; manufacturing and distribution activities routinely cross international borders. With the globalization of both sources of production and markets, firms must rethink production planning strategies. One issue explored in this chapter was smoothing of production plans over time; costs of increasing or decreasing workforce levels (and, hence, production levels) play a major role in the optimization of any aggregate plan. When formulating global production strategies, other smoothing issues arise. Exchange rates, costs of direct labor, and tax structure are just some of the differences among countries that must be factored into a global strategy.

Why the increased interest in global operations? In short, cost and competitiveness. According to McGrath and Bequillard (1989):

"The benefits of a properly executed international manufacturing strategy can be very substantial. A well developed strategy can have a direct impact on the financial performance of a firm and ultimately be reflected in increased profitability. In the electronic industry, there are examples of companies attributing 5% to 15% reduction in cost of goods sold, 10% to 20% decrease in workforce, and 50% to 100% increase in inventory turnover to their internationalization of manufacturing."

Cohen et al. (1989) outline some of the issues that a firm must consider when planning production levels on a worldwide basis. These include:

- In order to achieve the kinds of economies of scale required to be competitive today, multinational plants and vendors must be managed as a global system.
- Duties and tariffs are based on material flows. Their impact must be factored into decisions regarding shipment of raw material, intermediate product, and finished product across national boundaries.
- Exchange rates fluctuate randomly and affect production costs and pricing decisions in countries where the product is produced and sold.
- Corporate tax rates vary widely from one country to another.
- Global sourcing must take into account longer lead times, lower unit costs, and access to new technologies.

3.11 PRACTICAL CONSIDERATIONS

Aggregate planning can be a valuable aid in planning production and workforce levels for a company, and provides a means of absorbing demand fluctuations by smoothing workforce and production levels. There are a number of advantages for planning on the aggregate level over the detail level. One is that the cost of preparing forecasts and determining productivity and cost parameters on an individual item basis can be prohibitive. The cost of data collection and input is probably a more significant drawback of large-scale mathematical programming formulations of detailed production plans than is the cost of actually performing the computations. A second advantage of aggregate planning is the relative improvement in forecast accuracy that can be achieved by aggregating items. As
we saw in Chapter 2 on forecasting, aggregate planning forecasts are generally more accurate than individual forecasts. Finally, an aggregate planning framework allows the manager to see the "big picture" and not be unduly influenced by specifics.

With all these advantages, one would think that aggregate planning would have an important place in the planning of the production activities of many companies. However, this does not appear to be the case. There are a number of reasons for the lack of interest in aggregate planning methodology. First is the difficulty of properly defining an aggregate unit of production. There appears to be no simple way of specifying how individual items should be aggregated that will work in all situations. Second, once an aggregating scheme is developed, cost estimates and demand forecasts for aggregate units are required. It is difficult enough to obtain accurate cost and demand information for real units. Obtaining such information on an aggregate basis could be considerably more difficult, depending on the aggregate scheme that is used. Third, aggregate planning models rarely reflect the political and operational realities of the environment in which the company is operating. Assuming that workforce levels can be changed easily is probably not very realistic for most companies. Finally, as Silver and Peterson (1985) suggest, managers do not want to rely on a mathematical model for answers to the extremely sensitive and important issues that are addressed in this analysis.

Another issue is whether or not the goals of aggregate planning analysis can be achieved through methods other than those discussed in this chapter. Schwartz and Johnson (1978) claim that the cost savings achieved using a linear decision rule for aggregate planning could be obtained by improved management of the aggregate inventory alone. They substantiate their hypothesis using the data reported in Holt, Modigliani, Muth, and Simon (1960), and show that for the case of the paint company virtually all the cost savings of the linear decision rule were a result of increasing the buffer inventory and not of making major changes in the size of the labor force. This single comparison does not support the authors' hypothesis in general, but suggests that better inventory management, using the methods to be discussed in Chapters 4 and 5, could return a large portion of the benefits that a company might realize with aggregate planning.

Even with this long list of disadvantages, the mathematical models discussed in this chapter can, and should, serve as an aid to production planners. Although optimal solutions to a mathematical model may not be true optimal solutions to the problem that they address, they do provide insights and could reveal alternatives that would not otherwise be evident.

### 3.12 HISTORICAL NOTES

The aggregate planning problem was conceived in an important series of papers that appeared in the mid-1950s. The first, Holt, Modigliani, and Simon (1955), discussed the structure of the problem and introduced the quadratic cost approach, and the later study of Holt, Modigliani, and Muth (1956) concentrated on the computational aspects of the model. A complete description of the method and its application to production planning for a paint company is presented in Holt, Modigliani, Muth, and Simon (1960).

That production planning problems could be formulated as linear programs appears to have been known in the early 1950s. Bowman (1956) discussed the use of a transportation model for production planning. The particular linear programming formulation of the aggregate planning problem discussed in Section 3.5 is essentially the same as the one developed by Hansmann and Hess (1960). Other linear programming formulations of the production planning problem generally involve multiple products or multiple cost structures (see, for example, Newson, 1975a and 1975b).

More recent work on the aggregate planning problem has focused on aggregation and disaggregation issues (Axelstein, 1981; Bitrin and Hax, 1981; and Zoller, 1971), the incorporation of learning curves into linear decision rules (Ebert, 1976), extensions to allow for multiple products (Bergstrom and Smith, 1970), and inclusion of marketing and/or financial variables (Dunham and Schramm, 1972, and Leitch, 1974). Taubert (1968) considers a technique he refers to as the search decision rule. The method requires developing a computer simulation model of the system and searching the response surface using standard search techniques to obtain a solution. Another approach, which was described in detail in Buffa and Taubert (1972), gives results that are comparable to those of Holt, Modigliani, Muth, and Simon (1960) for the case of the paint company.

Kamien and Li (1990) developed a mathematical model to examine the effects of subcontracting on aggregate planning decisions. The authors show that under certain circumstances it is preferred to producing in-house, and provides an additional means of smoothing production and workforce levels.

### 3.13 Summary

Determining optimal production levels for all products produced by a large firm can be an enormous undertaking. Aggregate planning addresses this problem by assuming that individual items can be grouped together. However, finding an effective aggregating scheme can be difficult. The aggregating scheme chosen must be consistent with the structure of the organization and the nature of the products produced. One particular aggregating scheme that has been suggested is to group items, families, and types. Items or stockkeeping units (SKUs), representing the lowest level of detail, would be identified by separate part numbers. Families are groups of items that share a common manufacturing setup, and types are natural groupings of families. This particular aggregating scheme is fairly general, but there is no guaranteed that it will work in every application.

In aggregate planning one assumes deterministic demand. Demand forecasts over a specified planning horizon are required input. This assumption is not made for the sake of realism, but to allow the analysis to focus on the changes in the demand that are systematic rather than random. The goal of the analysis is to determine the number of workers that should be employed each period and the number of aggregate units that should be produced each period.

The objective is to minimize costs of production, payroll, holding, and changing the size of the workforce. The costs of making changes are generally referred to as smoothing costs. Most of the aggregate planning models discussed in this chapter assume that all the costs are linear functions. This means that the cost of hiring an additional worker is constant. Thus, the cost of hiring the previous worker, and the cost of holding an additional unit of inventory is the same as the cost of holding the previous unit of inventory. This assumption is probably a reasonable approximation for most real systems. It is unlikely that the primary problem with applying aggregate planning methodology to a real situation will be that the shape of the cost functions is incorrect; it is more likely that the primary difficulty will be in correctly estimating the costs and other required input information.

One can find approximate solutions to aggregate planning problems by graphical methods. The most common solution is the graph of cumulative production. This graph shows the cumulative production schedule that satisfies demand if the graph of cumulative production lies above the graph of cumulative demand. As long as all costs are linear, optimal solutions can be found by linear programming. Because solutions of linear programming...
problems may be fractions, some or all of the variables must be rounded to integers. The best way to round the variables is not always obvious. Once an optimal solution is found, its appropriateness must be considered in the context of the planning problem, as there are intangible costs and constraints that might be difficult or impossible to include directly in the problem formulation.

Linear programming also can be used to determine optimal production levels at the individual item level, but the size of the resulting problem and the input data requirements might make such an approach unrealistic for large companies. However, detailed linear programming formulations could be used at the family level rather than the individual item level for large firms.

Early work by Holt, Modigliani, Muth, and Simon (1960) required that the cost functions be quadratic functions. With quadratic costs, the form of the resulting optimal policy is a simple linear function, known as the linear decision rule. However, the assumption of quadratic cost functions is probably too restrictive to be accurate for most real problems.

Bowman's management coefficients model fits a reasonable mathematical model of the system to the actual decisions made by management over a period of time. This approach has the advantage of incorporating management's intuition into the model, and encourages management personnel to be consistent in future decision making.

Aggregate plans will be of little use to the firm if they cannot be coordinated with detailed item schedules (i.e., the master production schedule). The problem of disaggregating aggregate plans is a difficult one, but one that must be addressed if the aggregate plans are to have value to the firm. There have been some mathematical programming formulations of the disaggregating problem suggested in the literature, but these disaggregation schemes have yet to be proven in practice.

### Additional Problems on Aggregate Planning

29. An aggregate planning model is being considered for the following applications. Suggest an aggregate measure of production and discuss the difficulties of applying aggregate planning in each case.

- a. Planning for the size of the faculty in a university.
- b. Determining workforce requirements for a travel agency.
- c. Planning workforce and production levels in a fish-processing plant that produces canned sardines, anchovies, kippers, and smoked oysters.

30. A plant manager employed by a national producer of kitchen products is planning workforce levels for the next three months. An aggregate unit of production requires an average of four labor hours. Forecast demands for the three-month horizon are as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Workdays</th>
<th>Forecasted Demand (in aggregate units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>23</td>
<td>2,400</td>
</tr>
<tr>
<td>August</td>
<td>16</td>
<td>3,000</td>
</tr>
<tr>
<td>September</td>
<td>20</td>
<td>800</td>
</tr>
</tbody>
</table>

Assume that a normal workday is eight hours. Hiring costs are $500 per worker and firing costs are $1,000 per worker. Holding costs are 40 cents per aggregate unit held per month. Shortages are not permitted. Assume that the ending inventory for June was 600 and the manager wishes to have at least 800 units on hand at the end of September. The June workforce level is 30 workers.

- a. Find the minimum constant workforce plan for the three months and the total hiring, firing, and holding costs of that plan.
- b. Draw a graph of the cumulative net demand and use that graph to suggest a more efficient plan that makes one change in workforce levels during the planning horizon. Find the cost of the plan and compare it to the plan considered in part (a).

Assuming that this plan is more cost effective, comment on why.

31. A local firm manufactures children's toys. The projected demand over the next four months for one particular model of toy robot is

<table>
<thead>
<tr>
<th>Month</th>
<th>Workdays</th>
<th>Forecasted Demand (in aggregate units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>23</td>
<td>3,625</td>
</tr>
<tr>
<td>August</td>
<td>16</td>
<td>7,245</td>
</tr>
<tr>
<td>September</td>
<td>20</td>
<td>2,770</td>
</tr>
<tr>
<td>October</td>
<td>22</td>
<td>4,440</td>
</tr>
</tbody>
</table>

Assume that a normal workday is eight hours. Hiring costs are $350 per worker and firing costs (including severance pay) are $850 per worker. Holding costs are 40 cents per aggregate unit held per month. Assume that it requires an average of 1 hour and 40 minutes for one worker to assemble one toy. Shortages are not permitted. Assume that the ending inventory for June was 600 of these toys and the manager wishes to have at least 800 units on hand at the end of October. Assume that the current workforce level is 35 workers.

- a. Find the minimum constant workforce plan for the four months and the total hiring, firing, and holding costs of that plan.
- b. Determine the plan that changes the workforce level each month to most closely match the demand and the cost of that plan.

32. The Paris Paint Company is in the process of planning labor force requirements and production levels for the next four quarters. The marketing department has provided the following forecasts of demand for Paris Paint over the next year:

<table>
<thead>
<tr>
<th>Quarter</th>
<th>Demand Forecast (in thousands of gallons)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>380</td>
</tr>
<tr>
<td>2</td>
<td>630</td>
</tr>
<tr>
<td>3</td>
<td>2,20</td>
</tr>
<tr>
<td>4</td>
<td>160</td>
</tr>
</tbody>
</table>

Assume that there are currently 280 employees with the company. Employees are hired for at least one full quarter. Hiring costs amount to $1,200 per employee and firing costs are $2,500 per employee. Inventory costs are $1 per gallon per quarter. It is estimated that one worker produces 3,000 gallons of paint each quarter.
Assume that Paris currently has 80,000 gallons of paint in inventory and would like to end the year with an inventory of at least 20,000 gallons.

a. Determine the minimum constant workforce plan for Paris Paint and the cost of the plan. Assume that stock-outs are not allowed.

b. Determine the zero inventory plan that hires and fires workers each quarter to match demand as closely as possible and the cost of that plan.

c. If Paris were able to back-order excess demand at a cost of $2 per gallon per quarter, determine a minimum constant workforce plan that holds less inventory than the plan you found in part (a), but incurs stock-outs in quarter 2. Determine the cost of the new plan.

33. Consider the problem of Paris Paint presented in Problem 32. Suppose that the plant has the capacity to employ a maximum of 370 workers. Suppose that regular-time employee costs are $12.50 per hour. Assume seven-hour days, five-day weeks, and four-week months. Overtime is paid at a rate of $20 per hour of paint produced. Overtime is limited to three hours per employee per day, and no more than 100,000 gallons can be subcontracted in any quarter. Determine a policy that meets the demand, and the cost of that policy. (Assume stock-outs are not allowed.)

34. a. Formulate Problem 32 as a linear program. Assume that stock-outs are not allowed.

b. Solve the linear program. Round the variables and determine the cost of the resulting plan.

35. a. Formulate Problem 33 as a linear program.

b. Solve the linear program. Round the variables and determine the cost of the resulting plan.

36. The Mr. Meadows Cookie Company can obtain accurate forecasts for 12 months based on firm orders. These forecasts and the number of workdays per month are as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Demand Forecast (in thousands of cookies)</th>
<th>Workdays</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>850</td>
<td>26</td>
</tr>
<tr>
<td>2</td>
<td>1,260</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>510</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>980</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>770</td>
<td>22</td>
</tr>
<tr>
<td>6</td>
<td>850</td>
<td>23</td>
</tr>
<tr>
<td>7</td>
<td>1,050</td>
<td>14</td>
</tr>
<tr>
<td>8</td>
<td>1,550</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>1,350</td>
<td>23</td>
</tr>
<tr>
<td>10</td>
<td>1,000</td>
<td>24</td>
</tr>
<tr>
<td>11</td>
<td>970</td>
<td>21</td>
</tr>
<tr>
<td>12</td>
<td>880</td>
<td>13</td>
</tr>
</tbody>
</table>

During a 46-day period when there were 120 workers, the firm produced 1,700,000 cookies. Assume that there are 100 workers employed at the beginning of month 1 and zero starting inventory.

a. Find the minimum constant workforce needed to meet monthly demand.

b. Assume \( c_1 = 0.10 \) per cookie per month, \( c_2 = 100 \), and \( c_3 = 200 \). Add columns that give the cumulative on-hand inventory and inventory cost. What is the total cost of the constant workforce plan?

c. Construct a plan that changes the level of the workforce to meet monthly demand as closely as possible. In designing the logic for your calculations, be sure that inventory does not go negative in any month. Determine the cost of this plan.

37. The Yeasty Brewing Company produces a popular local beer known as Iron Stomach. Beer sales are somewhat seasonal, and Yeasty is planning its production and workforce levels on March 31 for the next six months. The demand forecasts are as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Production Days</th>
<th>Forecasted Demand (in hundreds of cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>April</td>
<td>11</td>
<td>85</td>
</tr>
<tr>
<td>May</td>
<td>22</td>
<td>93</td>
</tr>
<tr>
<td>June</td>
<td>20</td>
<td>122</td>
</tr>
<tr>
<td>July</td>
<td>23</td>
<td>176</td>
</tr>
<tr>
<td>August</td>
<td>16</td>
<td>140</td>
</tr>
<tr>
<td>September</td>
<td>20</td>
<td>63</td>
</tr>
</tbody>
</table>

As of March 31, Yeasty had 80 workers on the payroll. Over a period of 26 working days when there were 100 workers on the payroll, Yeasty produced 12,000 cases of beer. The cost to hire each worker is $125 and the cost of laying off each worker is $300. Holding costs amount to 75 cents per case per month. As of March 31, Yeasty expects to have 4,500 cases of beer in stock, and it wants to maintain a minimum buffer inventory of 1,000 cases each month. It plans to start October with 3,000 cases on hand.

a. Based on this information, find the minimum constant workforce plan for Yeasty over the six months, and determine hiring, firing, and holding costs associated with that plan.

b. Suppose that it takes one month to train a new worker. How will that affect your solution?

c. Suppose that the maximum number of workers that the company can expect to be able to hire in one month is 10. How will that affect your solution to part (a)?

d. Determine the zero inventory plan and the cost of that plan. (You may ignore the conditions in parts (b) and (c).)

38. a. Formulate the problem of planning Yeasty's production levels, as described in Problem 37, as a linear program. (You may ignore the conditions in parts (b) and (c).)

b. Solve the resulting linear program. Round the appropriate variables and determine the cost of your solution.

c. Suppose Yeasty does not wish to have any workers. What is the optimal plan subject to this constraint?

39. A local canning company sells canned vegetables to a supermarket chain in the Minneapolis area. A typical case of canned vegetables requires an average of 0.2 day of labor to produce. The aggregate inventory on hand at the end of June is
The demand for the vegetables can be accurately predicted for about 18 months based on orders received by the firm. The predicted demands for the next 18 months are as follows:

<table>
<thead>
<tr>
<th>Month</th>
<th>Actual Demand (hundreds of cases)</th>
<th>New Forecast (hundreds of cases)</th>
</tr>
</thead>
<tbody>
<tr>
<td>July</td>
<td>23</td>
<td>29</td>
</tr>
<tr>
<td>August</td>
<td>28</td>
<td>33</td>
</tr>
<tr>
<td>September</td>
<td>42</td>
<td>31</td>
</tr>
<tr>
<td>October</td>
<td>26</td>
<td>20</td>
</tr>
<tr>
<td>November</td>
<td>29</td>
<td>16</td>
</tr>
<tr>
<td>December</td>
<td>58</td>
<td>33</td>
</tr>
<tr>
<td>January</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td>February</td>
<td>17</td>
<td>28</td>
</tr>
<tr>
<td>March</td>
<td>25</td>
<td>50</td>
</tr>
</tbody>
</table>

The firm currently has 25 workers. The cost of hiring and training a new worker is $1,000, and the cost to lay off one worker is $1,500. The firm also incurs a cost of $2.80 for each case of vegetables stored for a month. They would like to have 1,500 cases in inventory at the end of the 18-month planning horizon.

a. Develop a spreadsheet to find the minimum constant workforce aggregate plan and determine the total cost of that plan.
b. Develop a spreadsheet to find a plan that hires and fires workers monthly in order to minimize inventory costs. Determine the total cost of that plan as well.
c. Graph the cumulative net demand, and using the graph, find a plan that changes the workforce level exactly once during the 18 months and has a lower cost than either of the plans considered in parts (a) and (b).

40. Consider the linear production rule of Holt, Modigliani, Muth, and Simon, appearing in Problem 24. Design a spreadsheet to compute $P_t$ and $H_t$ from demand forecasts. In particular, suppose that at the end of December 1995 the workforce was at 180 and the inventory was at 45 aggregate units.

- Using the forecasts of demand for the months January through December 1996 given in Problem 24, find the recommended levels of aggregate production and workforce size for January.
- Design your spreadsheet so that it can efficiently update recommended levels as demands are realized. In particular, suppose that the demand for January was actually 175 and the forecast for January 1997 was 120. What are the recommended levels of aggregate production and workforce size for February? (Hint: Store the demand forecasts and coefficients in separate columns and modify the computing formula as new data are appended to the old.)
- The actual realizations of demand and new forecasts of demand for the remainder of 1996 are given in the following table. Determine updated values for $P_t$ and $H_t$ each month.

### Glossary of Notation for Chapter 3

- $a$ = Smoothing constant for production and demand used in Bowman's model.
- $B$ = Smoothing constant for inventory used in Bowman's model.
- $c_P$ = Cost of firing one worker.
- $c_H$ = Cost of hiring one worker.
- $c_I$ = Cost of holding one unit of stock for one period.
- $c_D$ = Cost of producing one unit on overtime.
- $c_P$ = Penalty cost for back orders.
- $c_R$ = Cost of producing one unit on regular time.
- $c_S$ = Cost to subcontract one unit of production.
- $c_T$ = Idle cost per unit of production.
- $D_t$ = Forecast of demand in period $t$.
- $F_t$ = Number of workers fired in period $t$.
- $H_t$ = Number of workers hired in period $t$.
- $I_t$ = Inventory level in period $t$.
- $K_t$ = Number of aggregate units produced by one worker in one day.
- $A_t$ = Annual demand for family (refer to Section 3.9).
- $N_t$ = Number of production days in period $t$.
- $O_t$ = Overtime production in units.
- $P_t$ = Production level in period $t$.
- $S_t$ = Number of units subcontracted from outside.
- $T$ = Number of periods in the planning horizon.
- $U_t$ = Worker idle time in units ("undertime").
- $W_t$ = Workforce level in period $t$.
1. Introduction

Linear programming is a mathematical technique for solving a broad class of optimization problems. These problems require maximizing or minimizing a linear function of $n$ real variables subject to $m$ constraints. One can formulate and solve a large number of real problems with linear programming. A partial list includes:

1. Scheduling of personnel.
2. Several varieties of blending problems including cattle feed, sausages, ice cream, and steel making.
3. Inventory control and production planning.
4. Distribution and logistics problems.
5. Assignment problems.

Problems with thousands of variables and thousands of constraints are easily solvable on computers today. Linear programming was developed to solve logistics problems during World War II. George Dantzig, a mathematician employed by the RAND Corporation at the time, developed a solution procedure he labeled the Simplex Method. The method turned out to be so efficient for solving large problems quickly was a surprise even to its developer. That was because it could not be applied to problems using electronic computers, established linear programming as an important tool in logistics management. The success of linear programming in industry spurred the development of the method in the discipline of operations research and management science. The Simplex Method has withstood the test of time. Only in recent years has another method been developed that potentially could be more efficient than the Simplex Method. For solving very large, specially structured, linear programs, this method, known as Karmarkar’s Algorithm, was named after the Bell Labs mathematician who conceived it.

In Section S1.2 we consider a typical manufacturing problem that we formulate and solve using linear programming. Later we explore how one solves small problems (if it is, having exactly two decision variables) graphically, and how one solves large problems using a computer.

S1.2 A Prototype Linear Programming Problem

Sidneyville manufactures household and commercial furnishings. The Office Division produces two desks, rolltop and regular. Sidneyville constructs the desks in its plant outside Medford, Oregon, from a selection of woods. The woods are cut to a uniform thickness of 1 in. For this reason, one measures the wood in units of square feet. One rolltop desk requires 8 square feet of pine, 4 square feet of cedar, and 15 square feet of maple. One regular desk requires...